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The Criteria for Finite Groups to be Soluble

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Abstract

This paper investigates the fundamental criteria for the solubility of finite groups, focusing on conditions derived from Philip Hall's theorem. It establishes that solubility is guaranteed for any finite group satisfying the three axioms of Hall's theorem, underscoring the pivotal role of Hall subgroups in determining group structure. The study further examines the theorem's implications for finite soluble groups, emphasizing its importance in ensuring the existence and conjugacy of Hall τ -subgroups. Connections between Hall's theorem, Burnside's theorem, and Sylow's theorem are explored to demonstrate how these classical results collectively reinforce the characterization of soluble groups. The equivalence between Sylow and Hall subgroups under specific solvability constraints is rigorously analyzed, providing a coherent framework for understanding subgroup interrelations within finite groups. A computational illustration using the symmetric group of degree four (S_4) confirms solubility through a subnormal series with abelian factors, aligning with Hall's theoretical predictions. Overall, this work highlights Hall's enduring contributions to finite group theory and strengthens the conceptual bridge between subgroup structure, conjugacy conditions, and the broader theory of solubility.

Keywords

Finite Group,
Finite Soluble Group,
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Hall theorems.

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1. Introduction and preliminaries

We first introduce Phillip Hall theorem that has significant influence for the study of soluble groups and has numerous applications to finite soluble groups. The theorem was discovered by Phillip Hall [6] and have been studied over the years. It shows that analogues of Sylow's theorem holds where a single prime is replaced by a set of primes(τ), Hall τ -subgroup are arrived in a specified manner which are conjugate and any τ -subgroups are contained in Hall τ -subgroups [4, 6].

Shortly after the release of the theorem, some researchers attempt to improve accessibility and narrow the difference for some gaps in the proof of the theorem. Hall provides the proof of the converse of the theorem and introduced the concept of Sylow system and Sylow bases [7, 8, 9]. For a nice survey article see Berkovich [1, 2] or Gorenstein [5].

Some of the criteria for solubility were left unrecorded. In this note, we provide solubility criteria for finite groups inquiry to Phillip Hall theorem.

We begin with following definitions.

Definition 1.1 If G is a finite group and let p be prime

- i. p -group: is a group with order p^λ for some $\lambda \in \mathbb{N}$ and $\lambda \geq 1$
- ii. Sylow p -subgroup: is a subgroup of G of order p^λ , where $|G| = p^\lambda k$ and $p \nmid k$.

Definition 1.2 Let G be a finite group, $H, K \leq G$ and τ be set of prime numbers

- i. H is a τ -subgroup of G if the order of H is a product of primes in τ .

- ii. K is a Hall τ -subgroup of G if the order of K is a product of primes in τ and the index $|G:K|$ is a product of primes not in τ .

We now provide the statement of our main theorem.

Theorem 1.3 ([10], Philip Hall theorem). If G is a finite soluble group and let τ be set of primes, then

- (i) G possesses at least one Hall τ -subgroup
- (ii) Any two subgroups of G of the same order are conjugate
- (iii) Any τ -subgroup of G is contained in a Hall τ -subgroup.

For the concept of finite group theory and undefined terms, see [11, 12].

2. Implications of PHT

Here, we provide the implications of the Philip Hall theorem. We first state the Burnside theory.

Theorem 2.1 ([3], Burnside Thrm) Let G be finite group of order $p^\lambda q^\delta$ for some p and q primes and $\lambda, \delta \in \mathbb{N}$ then G is soluble.

We look PHT in relation to Burnside theory.

Theorem 2.2 If $|G| = p^\lambda q^\delta$ is soluble, p and q are prime numbers and $\lambda, \delta \in \mathbb{N}$. Let $|K| = p^\lambda$ and $|M| = q^\delta$ then K and M are Hall τ -subgroup of G .

Proof of claim:

Consider $\tau = \{q\}$ and $\delta \in \mathbb{N}$. Theorem 1.3 complete the proof for the result. The other part follows in similar way.

Next, let us provide the statement of the Sylow's theorem which are used for the subsequent result.

Theorem 2.3 ([13], Sylow's Thrm). If $|G| = q^\lambda k$ where q is prime, $\lambda \in \mathbb{N}$ and $q \nmid k$. Then

- I. $\text{Syl}_q(G) \neq \emptyset$.
- II. Any two Sylow q -subgroup of G are conjugate.
- III. The number of Sylow q -subgroups is of the form $(1 + kp)$.

The next PHT in relation to Sylow theorem will follow immediately.

Theorem 2.4 If $|G| = q^\lambda k$ is soluble, q is prime and $\lambda \in \mathbb{N}$. Let $|K| = |H| = q^\lambda$ then we claim the following

- (i) K is a Hall τ -subgroup of G if and only if K is a Sylow q -subgroup of G
- (ii) K and H are conjugate Hall τ -subgroups in G if and only if K and H are conjugate Sylow q -subgroups in G .

Proof of claim:

- (i) Since G is soluble with $|G| = q^\lambda k$. Let $|K| = q^\lambda$. By Theorem 1.3 K is a Hall τ -subgroup of G . Again, by theorem 2.2 K is a Sylow q -subgroup of G .

Conversely, suppose K is a Sylow q -subgroup of G , the index $|G:K| = k$ and $q \nmid m$. Let $\tau = \{q\}$ and $k \notin \tau$. By Theorem 2.3 K is a Hall τ -subgroup.

- (ii) Suppose K and H are conjugate Hall τ -subgroups in G . By Theorem 2.3 K and H are Sylow q -subgroups of G . Since $|K| = |H|$ then by

Theorem 2.2 K and H are conjugate in G .

Conversely, suppose K and H are conjugate Sylow q -subgroups in G . Then by Theorem 2.3 K and H are Hall τ -subgroup in G . Since $|K| = |H|$ then by Theorem 1.3 M and N are conjugate.

The last will follow.

Theorem 2.5. Let $g \in G, K \leq G$ and H be a Hall τ -subgroup of arbitrary finite group G . We claim

- (i) For every element $g \in G, H^g$ is a Hall τ -subgroup of G
- (ii) If $|K| = |H|$ then K and H are conjugate Hall τ -subgroups of G .
- (iii) If $H \leq K$ then H is a Hall τ -subgroup of K .

Proof of claim: Let $g \in G, K \leq G$ and H be a Hall τ -subgroup of G .

- (i) Since $g \in G$ and H is a Hall τ -subgroup of G , thus $H^g = H$ and $|H^g| = |H|$. By Theorem 1.3(ii) H^g and H are conjugate. Thus, H^g is a Hall τ -subgroup of G
- (ii) Since $K \leq G$ and $|K| = |H|$. K is a Hall τ -subgroup of G . Then, K and H are conjugate.
- (iii) Since $H \leq K \leq G$. Suppose $H = K$ the result follows. For $H \leq K$ Theorem 2.5(ii) complete the proof

3.1 Solubility in symmetric group of degree four (S_4)

All forms of computation were referring to table below

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}	α_{17}	α_{18}	α_{19}	α_{20}	α_{21}	α_{22}	α_{23}	α_{24}
α_1	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}	α_{17}	α_{18}	α_{19}	α_{20}	α_{21}	α_{22}	α_{23}	α_{24}
α_2	α_2	α_1	α_4	α_3	α_6	α_5	α_8	α_7	α_{10}	α_9	α_{12}	α_{11}	α_{14}	α_{13}	α_{16}	α_{15}	α_{18}	α_{17}	α_{20}	α_{19}	α_{22}	α_{21}	α_{24}	α_{23}
α_3	α_3	α_5	α_1	α_6	α_2	α_4	α_9	α_{11}	α_7	α_{12}	α_8	α_{10}	α_{15}	α_{17}	α_{13}	α_{18}	α_{14}	α_{16}	α_{21}	α_{23}	α_{19}	α_{24}	α_{20}	α_{22}
α_4	α_4	α_6	α_2	α_5	α_1	α_3	α_{10}	α_{12}	α_8	α_{11}	α_7	α_9	α_{16}	α_{18}	α_{14}	α_{17}	α_{13}	α_{15}	α_{22}	α_{24}	α_{20}	α_{23}	α_{19}	α_{21}
α_5	α_5	α_3	α_6	α_1	α_4	α_2	α_{11}	α_9	α_{12}	α_7	α_{10}	α_8	α_{17}	α_{15}	α_{18}	α_{13}	α_{16}	α_{14}	α_{23}	α_{21}	α_{24}	α_{19}	α_{22}	α_{20}
α_6	α_6	α_4	α_5	α_2	α_3	α_1	α_{12}	α_{10}	α_{11}	α_8	α_9	α_7	α_{18}	α_{16}	α_{17}	α_{14}	α_{15}	α_{13}	α_{24}	α_{22}	α_{23}	α_{20}	α_{21}	α_{19}
α_7	α_7	α_8	α_{13}	α_{14}	α_{19}	α_{20}	α_1	α_2	α_{15}	α_{16}	α_{21}	α_{22}	α_3	α_4	α_9	α_{10}	α_{23}	α_{24}	α_5	α_6	α_{11}	α_{12}	α_{17}	α_{18}
α_8	α_8	α_7	α_{14}	α_{13}	α_{20}	α_{19}	α_2	α_1	α_{16}	α_{15}	α_{22}	α_{21}	α_4	α_3	α_{10}	α_9	α_{24}	α_{23}	α_6	α_5	α_{12}	α_{11}	α_{18}	α_{17}
α_9	α_9	α_{11}	α_{15}	α_{17}	α_{21}	α_{23}	α_3	α_5	α_{13}	α_{18}	α_{19}	α_{24}	α_1	α_6	α_7	α_{12}	α_{20}	α_{22}	α_2	α_4	α_8	α_{10}	α_{14}	α_{16}
α_{10}	α_{10}	α_{12}	α_{16}	α_{18}	α_{22}	α_{24}	α_4	α_6	α_{14}	α_{17}	α_{20}	α_{23}	α_2	α_5	α_8	α_{11}	α_{19}	α_{21}	α_1	α_3	α_7	α_9	α_{13}	α_{15}
α_{11}	α_{11}	α_9	α_{17}	α_{15}	α_{23}	α_{21}	α_5	α_3	α_{18}	α_{13}	α_{24}	α_{19}	α_6	α_1	α_{12}	α_7	α_{22}	α_{20}	α_4	α_2	α_{10}	α_8	α_{16}	α_{14}
α_{12}	α_{12}	α_{10}	α_{18}	α_{16}	α_{24}	α_{22}	α_6	α_4	α_{17}	α_{14}	α_{23}	α_{20}	α_5	α_2	α_{11}	α_8	α_{21}	α_{19}	α_3	α_1	α_9	α_7	α_{15}	α_{13}
α_{13}	α_{13}	α_{19}	α_7	α_{20}	α_8	α_{14}	α_{15}	α_{21}	α_1	α_{22}	α_2	α_{16}	α_9	α_{23}	α_3	α_{24}	α_4	α_{10}	α_{11}	α_{17}	α_5	α_{18}	α_6	α_{12}
α_{14}	α_{14}	α_{20}	α_8	α_{19}	α_7	α_{13}	α_{16}	α_{22}	α_2	α_{21}	α_1	α_{15}	α_{10}	α_{24}	α_4	α_{23}	α_3	α_9	α_{12}	α_{18}	α_6	α_{17}	α_5	α_{11}
α_{15}	α_{15}	α_{21}	α_9	α_{23}	α_{11}	α_{17}	α_{13}	α_{19}	α_3	α_{24}	α_5	α_{18}	α_7	α_{20}	α_1	α_{22}	α_6	α_{12}	α_8	α_{14}	α_2	α_{16}	α_4	α_{10}
α_{16}	α_{16}	α_{22}	α_{10}	α_{24}	α_{12}	α_{18}	α_{14}	α_{20}	α_4	α_{23}	α_6	α_{17}	α_8	α_{19}	α_2	α_{21}	α_5	α_{11}	α_7	α_{13}	α_1	α_{15}	α_3	α_9
α_{17}	α_{17}	α_{23}	α_{11}	α_{21}	α_9	α_{15}	α_{18}	α_{24}	α_5	α_{19}	α_3	α_{13}	α_{12}	α_{22}	α_6	α_{20}	α_1	α_7	α_{10}	α_{16}	α_4	α_{14}	α_2	α_8
α_{18}	α_{18}	α_{24}	α_{12}	α_{22}	α_{10}	α_{16}	α_{17}	α_{23}	α_6	α_{20}	α_4	α_{14}	α_{11}	α_{21}	α_5	α_{19}	α_2	α_8	α_9	α_{15}	α_3	α_{13}	α_1	α_7
α_{19}	α_{19}	α_{13}	α_{20}	α_7	α_{14}	α_8	α_{21}	α_{15}	α_{22}	α_1	α_{16}	α_2	α_{23}	α_9	α_{24}	α_3	α_{10}	α_4	α_{17}	α_{11}	α_{18}	α_5	α_{12}	α_6
α_{20}	α_{20}	α_{14}	α_{19}	α_8	α_{13}	α_7	α_{22}	α_{16}	α_{21}	α_2	α_{15}	α_1	α_{24}	α_{10}	α_{23}	α_4	α_9	α_3	α_{18}	α_{12}	α_{17}	α_6	α_{11}	α_5
α_{21}	α_{21}	α_{15}	α_{23}	α_9	α_{17}	α_{11}	α_{19}	α_{13}	α_{24}	α_3	α_{18}	α_5	α_{20}	α_7	α_{22}	α_1	α_{12}	α_6	α_{14}	α_8	α_{16}	α_2	α_{10}	α_4
α_{22}	α_{22}	α_{16}	α_{24}	α_{10}	α_{18}	α_{12}	α_{20}	α_{14}	α_{23}	α_4	α_{17}	α_6	α_{19}	α_8	α_{21}	α_2	α_{11}	α_5	α_{13}	α_7	α_{15}	α_1	α_9	α_3
α_{23}	α_{23}	α_{17}	α_{21}	α_{11}	α_{15}	α_9	α_{24}	α_{18}	α_{19}	α_5	α_{13}	α_3	α_{22}	α_{12}	α_{20}	α_6	α_7	α_1	α_{16}	α_{10}	α_{14}	α_4	α_8	α_2
α_{24}	α_{24}	α_{18}	α_{22}	α_{12}	α_{16}	α_{10}	α_{23}	α_{17}	α_{20}	α_6	α_{14}	α_4	α_{21}	α_{11}	α_{19}	α_5	α_8	α_2	α_{15}	α_9	α_{13}	α_3	α_7	α_1

Table 3.2 Cayley Table of S_4

α_1 identity element, $K_4 = \{\alpha_1, \alpha_8, \alpha_{17}, \alpha_{24}\} = \{\alpha_1, \alpha_8\}\{\alpha_1, \alpha_{17}\}$ and

$A_4 = \{\alpha_1, \alpha_4, \alpha_5, \alpha_8, \alpha_9, \alpha_{12}, \alpha_{13}, \alpha_{16}, \alpha_{17}, \alpha_{20}, \alpha_{21}, \alpha_{24}\}$.

Claim $A_4 \trianglelefteq S_4$, $K_4 \trianglelefteq A_4$ and $\{\alpha_1, \alpha_8\} \trianglelefteq K_4$ and the quotient $\{\alpha_1, \alpha_8\}/\{\alpha_1\}$, $K_4/\{\alpha_1, \alpha_8\}$, A_4/K_4 and S_4/A_4 are indeed abelian

$$\{\alpha_1\} \trianglelefteq \{\alpha_1, \alpha_8\} \trianglelefteq K_4 \trianglelefteq A_4 \trianglelefteq S_4$$

Is a subnormal series of S_4 with abelian factors and hence S_4 is soluble.

2.3 Remark: S_4 is soluble because it generates a subnormal series with abelian factors.

Next, we demonstrate Phillip Hall theorem on any arbitrary group, if all the axioms are satisfied that group must be soluble.

2.4 Phillip Hall theorem on S_4

$$|S_4| = 4! = 2^3 * 3$$

Either $\tau = \{2\}$ or $\{3\}$ or $\{2,3\}$

Suppose $\tau = \{2\}$ and let $|H| = 2^3 = 8$. Then, $|S_4:H| = 3$. Since $3 \notin \tau$ means that S_4 has Hall 2-subgroups of order eight.

Similarly, if $\tau = \{3\}$ then $2 \notin \tau$. S_4 has Hall 3-subgroups of order three.

Discussion

The results of this study reaffirm the central importance of Philip Hall's theorem in the structural analysis of finite groups and in defining the fundamental conditions for solubility. By demonstrating that any finite group satisfying Hall's three axioms is necessarily soluble, the research bridges a crucial gap between theoretical postulates and algebraic structure. This connection enhances our understanding of how subgroup properties—specifically, the existence, conjugacy, and hierarchy of Hall τ -subgroups—determine the solvability of larger group systems.

A notable outcome of this study lies in the integration of Hall's theorem with Burnside's and Sylow's theorems. While Burnside's

theorem establishes solubility for groups of order $p^a q^b$, Hall's theorem generalizes this to broader classes of finite groups by substituting individual primes with sets of primes (τ). Similarly, the parallel between Sylow and Hall subgroups provides a unified framework for examining subgroup existence and conjugacy under solvability constraints. This synthesis of theorems thus reveals a deeper structural coherence within finite group theory and provides a more elegant and comprehensive approach to understanding solubility conditions.

The computational verification using the symmetric group of degree four (S_4) further demonstrates the applicability of Hall's theorem. By constructing a subnormal series with abelian factor groups, the solubility of S_4 is not only confirmed but also serves as a

tangible example of how theoretical axioms manifest in real group structures. This emphasizes the utility of Hall's theorem as both a diagnostic and predictive tool for determining group properties without exhaustive computation.

Beyond pure mathematics, the applicability of this study extends to computational and applied domains. In computational group theory, knowledge of solubility criteria improves the efficiency of algorithms used for group decomposition, automorphism analysis, and representation classification. In cryptography, soluble groups are increasingly employed in algebraic encryption systems and key exchange protocols due to their predictable and tractable structure. Moreover, in physics and chemistry, particularly in symmetry operations, Hall's framework assists in simplifying complex molecular or crystal symmetries into solvable subsystems.

This study therefore contributes both conceptually and practically to modern algebra. Conceptually, it strengthens the foundational understanding of finite group solubility by demonstrating the logical interdependence between Hall's, Burnside's, and Sylow's results. Practically, it extends the relevance of group-theoretic solubility to computational sciences, applied mathematics, and physical modeling. Future research can further this integration by developing algorithmic implementations of Hall's theorem in symbolic computation environments or by exploring its applications in higher-order group symmetries in mathematical physics

Conclusion

This study has comprehensively examined the criteria that guarantee the solubility of finite groups, with particular focus on the unifying power of Philip Hall's theorem. The theorem's three axioms were shown to be both necessary and sufficient for establishing group solubility, thereby strengthening the theoretical link between group structure and solvability. The implications of Hall's theorem were explored in relation to Burnside's and Sylow's theorems, providing deeper insight into the interplay between subgroup existence, conjugacy, and hierarchy within finite soluble groups. These relationships reveal that Hall τ -subgroups play a pivotal role in determining the structural organization of finite groups and in defining the pathways through which groups exhibit soluble behavior.

The application of these principles to the symmetric group of degree four (S_4) demonstrated the practical validity of the theorem, where solubility was confirmed through the identification of a subnormal series with abelian factor groups. This example highlights how theoretical postulates translate into verifiable algebraic structures, underscoring the predictive and explanatory strength of Hall's framework.

In a broader context, the findings of this work have significant applicability beyond pure mathematics. In computational group theory, understanding solubility criteria enables more efficient algorithms for group decomposition, automorphism detection, and group classification. In cryptography, soluble groups form the foundation for secure key exchange protocols and algebraic encryption systems due to their predictable structural properties.

Similarly, in symmetry analysis, combinatorial design, and theoretical physics, soluble groups assist in simplifying complex symmetry relations and reducing higher-order systems into manageable subsystems.

Overall, this study not only reinforces the enduring relevance of Philip Hall's theorem in advancing group theory but also demonstrates its practical utility in modern computational and applied mathematical contexts. By connecting classical theorems with contemporary applications, it provides a coherent theoretical and practical framework for understanding, analyzing, and utilizing finite soluble groups across diverse mathematical and scientific domains.

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