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## Modified Exponential Ratio Estimator Using Two Auxiliary Variables

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### Abstract

This study contributes to the growing body of literature on survey sampling by enhancing the efficiency of population mean estimation through the incorporation of auxiliary information. In sample survey methodology, the integration of auxiliary variables at the estimation stage has become a fundamental strategy for improving estimator precision relative to conventional unbiased estimators. Building on this premise, previous research has demonstrated that the use of a single auxiliary variable can significantly reduce estimation variance; more recent advancements have extended this framework to accommodate multiple auxiliary variables for further efficiency gains. In this research, two auxiliary variables are simultaneously utilized to develop a class of modified ratio-type estimators under a double sampling (two-phase sampling) scheme. The proposed estimator incorporates a system of weights—arbitrarily assigned to each auxiliary variable—to optimize their relative contributions during estimation. Furthermore, an exponential functional form is embedded within the estimator to attenuate the influence of extreme observations and mitigate the effect of outliers, thereby enhancing robustness. The theoretical properties of the proposed estimator are rigorously derived using first-order approximation techniques. Specifically, expressions for the Mean Square Error (MSE) are obtained and compared analytically with those of existing estimators in the literature. Conditions under which the proposed estimator outperforms conventional ratio and regression-type estimators are formally established.

To validate the theoretical findings, an empirical investigation based on simulated datasets was conducted. The simulation results demonstrate a consistent reduction in MSE for the proposed estimator relative to competing estimators, confirming its superior efficiency and stability. Overall, the study provides a robust and flexible estimation framework that leverages multiple auxiliary variables and exponential adjustment to achieve improved performance in practical survey applications.

### Keywords

*Finite population mean, Auxiliary variable, Weight, Bias and Mean square error.*

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## 1. Background of the study

The use of auxiliary information in survey sampling has been extensively investigated as a means of improving the efficiency of estimators for finite population parameters, particularly the population mean. A substantial body of literature exists on the development and modification of estimators involving a single auxiliary variable ( $x$ ) in relation to a study variable ( $y$ ) under both simple random sampling without replacement (SRSWOR) and with replacement (SRSWR) schemes. Seminal contributions in this domain include the works of Murthy (1967), Cochran (1977), Das and Tripathi (1978), Ahmed (1997), and Singh (1986), among others, who proposed numerous classes of estimators along with their statistical properties.

Auxiliary information is typically incorporated into estimation procedures through ratio, product, and regression-type estimators. The ratio estimator is most appropriate when a strong positive correlation exists between the study and auxiliary variables, whereas the product estimator is preferred under negative correlation structures (Olayiwola and Akintunde, 2020). The integration of such information at the estimation stage has been shown to significantly enhance the precision of population mean estimates (Olowookere et al., 2021). More recently, methodological advancements have extended these approaches to accommodate multiple auxiliary variables, thereby achieving further variance reduction.

Several researchers have proposed improved ratio-type estimators utilizing dual auxiliary variables under different sampling frameworks. For instance, Tailor and Chouhan (2014) developed a ratio-cum-product

exponential estimator for stratified random sampling, while Yadav et al. (2015a, 2015b), Verma et al. (2015), Shabbir and Gupta (2016), Jerajuddin and Kishun (2016), and Malik and Singh (2017) contributed to the development of efficient estimators for finite population means. In a two-phase sampling context, Olayiwola et al. (2020) introduced a class of ratio estimators based on dual auxiliary variables. Further extensions include chain ratio-type estimators under double sampling (Shabbir and Onyango, 2022; Singh and Nigam, 2022; Janbandhu et al., 2023) and recent developments by Kanwai et al. (2024) and Sher et al. (2024), which emphasize practical applications and improved estimator performance.

The theoretical foundation for multivariate auxiliary information was first established by Olkin (1958), who introduced a multivariate ratio estimator. This was subsequently extended by Raj (1965), Singh (1967, 1969), Srivastava (1965), and Upadhyaya et al. (1985), among others. Later contributions by Singh and Tailor (2005) incorporated correlation coefficients, while Gupta and Shabbir (2008) and Singh et al. (2009) proposed ratio-cum-product estimators. Swain (2013) and Sharma and Singh (2014) further expanded the class of ratio and product estimators.

In many practical survey situations, auxiliary information on more than one variable is readily available and can be effectively utilized to improve estimator precision. The incorporation of two auxiliary variables alongside the study variable enhances the efficiency and robustness of population mean estimators, particularly under complex sampling designs such as double sampling,

where complete auxiliary information may not be available at the initial stage.

This study extends the work of Singh and Nigam (2022) by developing a modified class of ratio-type estimators under a double sampling framework using two auxiliary variables. The proposed methodology systematically investigates different weighting structures, specifically examining cases where the sum of the weights assigned to the study and auxiliary variables  $((y, x, z))$  is equal to unity and less than unity. The weights are selected arbitrarily within the interval 0.1 to 0.9 (to one decimal place), allowing flexibility in modeling the relative influence of each variable. The proposed estimators are expected to contribute to the existing literature by

$$t_1 = \bar{y}_n \frac{\bar{x}_{n'}}{\bar{x}_n}, \quad (1)$$

$$MSE(t_1) = f_1 C_y^2 + f_1 (C_y^2 - 2R\rho_{yx} C_x C_y + R^2 C_x^2) \quad (2)$$

Singh and Vishwakarma (2007) suggested a ratio type exponential estimator after studying the properties of estimators by Bahl and Tutaja (1991). The ratio-type estimator is defined as;

$$t_4^{Re} = \bar{y} \exp\left(\frac{\bar{x}' - \bar{x}}{\bar{x}' + \bar{x}}\right) \quad (3)$$

$$MSE(t_4) = \bar{Y}^2 \left[ \frac{C_y^2}{n} + f_3 C_x^2 (1 - 2K_{yx}) + \frac{C_z^2}{n'} (1 - 2K_{yx}) \right] \quad (4)$$

Vishwakarma and Kumar (2014), motivated by the work of Singh and Choudhury (2012), suggested a ratio-product type estimators in two-phase sampling using two auxiliary variables.

$$t_5 = \bar{y} \left[ \alpha_1 \exp\left\{\frac{(\bar{x}'/\bar{z}')\bar{z} - \bar{x}}{(\bar{x}'/\bar{z}')\bar{z} + \bar{x}}\right\} + \alpha_2 \exp\left\{\frac{\bar{x} - (\bar{x}'/\bar{z}')\bar{z}}{\bar{x} + (\bar{x}'/\bar{z}')\bar{z}}\right\} \right] \quad (5)$$

$$MSE(t_5) = \bar{Y}^2 \left[ \begin{aligned} & f_1 C_y^2 + \frac{(2\alpha - 1)^2}{4} \{f_3 C_x^2 + f_2 C_z^2\} \\ & - (2\alpha - 1) \{f_3 \rho_{yx} C_x C_y + f_2 \rho_{yz} C_z C_y\} \end{aligned} \right] \quad (6)$$

Singh and Nigam (2022), suggested an estimator for population mean, defined as;

providing a more efficient and adaptable framework for estimating the population mean, particularly in situations where auxiliary information is partially available or subject to uncertainty.

## 2. Literature Review

If  $\psi$  population consisting of  $N$  units  $\{\psi_1, \psi_2, \psi_3 \dots \dots \dots \psi_N\}$ , and, let  $y$  and  $(x$  and  $z)$  be studying variable and auxiliary variables, respectively. Let  $y_i$  and  $\{x_i, z_i\}$  be the value of the studying variable  $y$  and auxiliary variable  $x$  and  $z$  on the  $i$ th unit  $U_i$  of the population  $U$ . For this purpose of double sampling procedure discussed below will be adopted. **Cochran (1940)** suggested a ratio estimator for double sampling scheme, defined as;

$$t_6 = \omega_0 \bar{y} + \omega_1 \bar{y} \left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha_1} + \omega_2 \bar{y} \left(\frac{\bar{z}}{\bar{Z}}\right)^{\alpha_2} \tag{7}$$

where  $(\omega_0, \omega_1, \omega_2)$  are weight, whose sum are equal to one  $(\alpha_1, \alpha_2)$  are design parameters. The constant  $(\alpha_1, \alpha_2)$ . Singh and Nigam (2022) adopted the method of Cramer row to obtain the unknown values of weight to each variable and this method cannot provide the sum of weight to be equal to one.

$$B(t_6) = \bar{Y} \left[ \begin{array}{l} \omega_0 \\ + \omega_1 \left\{ 1 + \left(\frac{1-f}{n}\right) C_x^2 [\alpha_1 + 2K_{xy} - 1] \frac{\alpha_1}{2} \right\} \\ + \omega_2 \left\{ 1 + \left(\frac{1-f}{n}\right) C_z^2 [\alpha_2 + 2K_{zy} - 1] \frac{\alpha_2}{2} \right\} - 1 \end{array} \right] \tag{8}$$

$$Bias(t_6) = \bar{Y} [\omega_0 + \omega_1 \tau_6 + \omega_2 \tau_7 - 1] \tag{9}$$

$$MSE(t_6) = \bar{Y}^2 \left[ 1 + \omega_0^2 \tau_0 + \omega_1^2 \tau_1 + \omega_2^2 \tau_2 + 2\omega_0 \omega_1 \tau_3 + 2\omega_0 \omega_2 \tau_4 + 2\omega_1 \omega_2 \tau_5 - 2\omega_0 - 2\omega_1 \tau_6 - 2\omega_2 \tau_7 \right] \tag{10}$$

where;  $\tau_0 = \left[ 1 + \left(\frac{1-f}{n}\right) C_y^2 \right]$

$$\tau_1 = \left[ 1 + \left(\frac{1-f}{n}\right) \{ C_y^2 + 4\alpha_1 \rho_{xy} C_x C_y + \alpha_1 (2\alpha_1 - 1) C_x^2 \} \right],$$

$$\tau_2 = \left[ 1 + \left(\frac{1-f}{n}\right) \{ C_y^2 + 4\alpha_2 \rho_{zy} C_z C_y + \alpha_2 (2\alpha_2 - 1) C_z^2 \} \right]$$

$$\tau_3 = \left[ 1 + \left(\frac{1-f}{n}\right) \left\{ C_y^2 + 2\alpha_1 \rho_{xy} C_x C_y + \frac{\alpha_1 (\alpha_1 - 1) C_x^2}{2} \right\} \right]$$

$$\tau_4 = \left[ 1 + \left(\frac{1-f}{n}\right) \left\{ C_y^2 + 2\alpha_2 \rho_{zy} C_z C_y + \frac{\alpha_2 (\alpha_2 - 1) C_z^2}{2} \right\} \right]$$

Janbandhu, et .al (2023), suggested a chain ratio type exponential estimator for finite population Mean in double sampling. The estimator defined as;

$$t_7 = \left(\frac{\bar{y}}{\bar{x}}\right) \bar{x}' \exp\left(\frac{\bar{z} - \bar{z}'}{\bar{z} + \bar{z}'}\right) \tag{11}$$

$$MSE(\widehat{t}_7) = \bar{Y}^2 \left[ f_1 C_y^2 + f_3 C_x^2 (1 - 2K_{xy}) + \frac{1}{4} f_2 C_z^2 (1 - 4K_{yz}) \right] \tag{12}$$

### 3. Materials and Methodology

Derivative of bias and mean square error of conventional ratio estimator

Cochran (1940), suggested a ratio estimator;

$$\widehat{Y}_R = \bar{y} \frac{\bar{X}}{\bar{x}} \tag{13}$$

Let  $\bar{y} = \bar{Y} (1 + e_0)$  and  $\bar{x} = \bar{X} (1 + e_1)$

$$= \frac{\bar{Y}(1+e_0)\bar{X}}{\bar{X}(1+e_1)} = (1+e_0)(1+e_1)^{-1}\bar{Y}$$

$$\hat{Y}_R = \bar{Y}(1+e_0)(1-e_1+e_1^2-\dots) \quad (14)$$

$$= \bar{Y}(1+e_0-e_1+e_1^2-e_0e_1+\dots) \quad (15)$$

So the estimated error of  $\hat{Y}_R$

$$\hat{Y}_R - \bar{Y} = \bar{Y}(1+e_0-e_1+e_1^2-e_0e_1+\dots) \quad (16)$$

$$\hat{Y}_R - \bar{Y} = \bar{Y}(e_0 - e_1) \text{ and } E(\hat{Y}_R - \bar{Y}) = 0. \quad (17)$$

$$\hat{Y}_R - \bar{Y} \approx \bar{Y}(1+e_0-e_1+e_1^2-e_0e_1+\dots) \quad (18)$$

Then the bias of  $\hat{Y}_R$  is given by

$$E(\hat{Y}_R - \bar{Y}) = \left(0 - 0 + \frac{f}{n}C_x^2 + \frac{f}{n}\rho_{yx}C_xC_y\right), \text{Bias}(\hat{Y}_R) = E(\hat{Y}_R - \bar{Y}) = \frac{f}{n}\bar{Y}(C_x - \rho_{yx}C_y).$$

To find the mean square error, consider

$$\begin{aligned} \text{MSE}(\hat{Y}_R) &= \bar{Y}^2 \left[ \frac{f}{n}C_x^2 + \frac{f}{n}C_y^2 - \frac{2f}{n}\rho_{yx}C_xC_y \right] \\ &= \bar{Y}^2 \frac{f}{n} [C_x^2 + C_y^2 - 2\rho_{yx}C_xC_y] \end{aligned} \quad (19)$$

#### 4. Proposed modified estimators

Motivated by the work of Singh and Nigam (2022), whose assigned the weight to the study variable and the auxiliary variables, slope of regression of study variable on auxiliary variable *estimator*  $k_1$  under double sampling given as:

$$k_1 = \omega_0\bar{y} + \omega_1\bar{y}\exp\left(\frac{\bar{x} + \beta_x}{\bar{x}' + \beta_x}\right)^{\alpha_1} + \omega_2\bar{y}\exp\left(\frac{\bar{z} + \beta_z}{\bar{z}' + \beta_z}\right)^{\alpha_2} \quad (20)$$

Where  $\bar{x}'$  and  $\bar{z}'$  are the sample mean of auxiliary variables X and Z respectively based on the first sample of size  $n'$  drawn from the population of size N using a simple random sample without replacement scheme. And,  $\bar{x}$  and  $\bar{z}$  are the sample mean of auxiliary variables X and Z respectively based on the second sample of size  $n$  drawn from the same population of size N using a simple random sample without replacement scheme.  $\omega_0$ ,  $\omega_1$  and  $\omega_2$ , are suitably chosen weights whose sum is equal to one, and  $(\alpha_1\alpha_2)$  are design parameters. And  $\beta_{yx}$  and  $\beta_{yz}$  are the slope of regression line between the study variable  $y$  and auxiliary variable  $x$ , and  $z$  respectively.

### 5. Properties of the Proposed Estimators

The biases and mean square errors of the suggested estimator, let assume the following conditions and define the error term as:

$$\begin{aligned} \bar{y} &= \bar{Y} (1 + e_0), \bar{x} = \bar{X} (1 + e_1) \text{ and } \bar{z} = \bar{Z} (1 + e_2), \bar{x}' = \bar{X} (1 + e_1') \text{ and } \bar{z}' = \bar{Z} (1 + e_2') \quad \text{such} \\ \text{that } E(e_0) &= E(e_1) = E(e_2) = E(e_1') = E(e_2') = 0, E(e_0^2) = \left(\frac{1}{n} - \frac{1}{N}\right) C_y^2 = f_1 C_y^2, \quad E \\ (e_1^2) &= \left(\frac{1}{n} - \frac{1}{N}\right) C_x^2 = f_1 C_x^2, E(e_2^2) = \left(\frac{1}{n} - \frac{1}{N}\right) C_z^2 = f_1 C_z^2, E(e_1')^2 = \left(\frac{1}{n'} - \frac{1}{N}\right) C_x^2 = f_2 C_x^2, \\ E(e_2')^2 &= \left(\frac{1}{n'} - \frac{1}{N}\right) C_z^2 = f_2 C_z^2, E(e_0 e_1) = \left(\frac{1}{n} - \frac{1}{N}\right) \rho_{yx} C_y C_x = f_1 \rho_{yx} C_y C_x, = f_1 K_{yx} C_x^2, \\ E(e_0 e_2) &= \left(\frac{1}{n} - \frac{1}{N}\right) \rho_{yz} C_y C_z = f_1 \rho_{yz} C_y C_z, = f_1 K_{yz} C_z^2, E(e_1 e_2) = \left(\frac{1}{n} - \frac{1}{N}\right) \rho_{xz} C_x C_z = f_1 \rho_{xz} C_x C_z \\ &= f_1 K_{xz} C_z^2, E(e_0 e_1') = \left(\frac{1}{n'} - \frac{1}{N}\right) \rho_{yx} C_y C_x = f_2 \rho_{yx} C_y C_x, = f_2 K_{yx} C_x^2, E(e_0 e_2') = \left(\frac{1}{n'} - \frac{1}{N}\right) \\ &\rho_{yz} C_y C_z = f_2 \rho_{yz} C_y C_z, = f_2 K_{yz} C_z^2, E(e_1 e_1') = \left(\frac{1}{n'} - \frac{1}{N}\right) C_x^2 = f_2 C_x^2, \end{aligned}$$

Derivative of Bias and mean square error of proposed estimator  $k_1$

$$k_1 = \omega_0 \bar{y} + \omega_1 \bar{y} \exp\left(\frac{\bar{x} + \beta_x}{\bar{x}' + \beta_x}\right)^{\alpha_1} + \omega_2 \bar{y} \exp\left(\frac{\bar{z} + \beta_z}{\bar{z}' + \beta_z}\right)^{\alpha_2}$$

Express the equation in term of error

$$k_1 = \omega_0 \bar{Y} (1 + e_0) + \omega_1 \bar{Y} (1 + e_0) \exp\left(\frac{\bar{X} (1 + e_1) + \beta_x}{\bar{Y} (1 + e_1') + \beta_x}\right)^{\alpha_1} + \omega_2 \bar{Y} (1 + e_0) \exp\left(\frac{\bar{Z} (1 + e_2) + C_z}{\bar{Z} (1 + e_2') + C_z}\right)^{\alpha_2} \quad (21)$$

$$\text{Let } \delta_{cx} = \frac{\bar{X}}{\bar{X} + \beta_x} \text{ and } \delta_{cz} = \frac{\bar{Z}}{\bar{Z} + \beta_z}$$

$$k_1 = \omega_0 \bar{Y} (1 + e_0) + \omega_1 \bar{Y} (1 + e_0) \exp\left(\frac{1 + \delta_{cx} e_1}{1 + \delta_{cx} e_1'}\right)^{\alpha_1} + \omega_2 \bar{Y} (1 + e_0) \exp\left(\frac{1 + \delta_{cz} e_2}{1 + \delta_{cz} e_2'}\right)^{\alpha_2} \quad (22)$$

$$\text{Let } \delta_{\beta_x} e_1 = e_1 \text{ and } \delta_{\beta_x} e_1' = e_1', \quad \text{Let } \delta_{\beta_z} e_2 = e_2 \text{ and } \delta_{\beta_z} e_2' = e_2'$$

$$\begin{aligned} k_1 &= \omega_0 \bar{Y} (1 + e_0) + \omega_1 \bar{Y} (1 + e_0) \exp[(1 + e_1)^{\alpha_1} (1 + e_1')^{-\alpha_1}] \\ &+ \omega_2 \bar{Y} (1 + e_0) \exp[(1 + e_2)^{\alpha_2} (1 + e_2')^{-\alpha_2}] \end{aligned} \quad (23)$$

Simplification and re-arrangement of the expression in term of power, we have

$$k_1 = \omega_0 \bar{Y} (1 + e_0) + \omega_1 \bar{Y} (1 + e_0) \exp \left[ \left( 1 + \alpha_1 e_1 - \alpha_1 e_1' - \alpha_1^2 e_1 e_1' + \frac{\alpha_1(\alpha_1-1)e_1^2}{2!} + \frac{\alpha_1(\alpha_1-1)e_1'^2}{2!} \right) \right. \\ \left. - \frac{\alpha_1^2(\alpha_2-1)e_1^2 e_1'}{2!} + \frac{\alpha_1^2(\alpha_2-1)e_1 e_1'^2}{2!} - \frac{\alpha_1^2(\alpha_1-1)^2 e_1^2 e_1'^2}{2!} \right] \\ + \omega_2 \bar{Y} (1 + e_0) \exp \left[ \left( 1 + \alpha_2 e_2 - \alpha_2 e_2' - \alpha_2^2 e_2 e_2' + \frac{\alpha_2(\alpha_2-1)e_2^2}{2!} + \frac{\alpha_2(\alpha_2-1)e_2'^2}{2!} \right) \right. \\ \left. - \frac{\alpha_2^2(\alpha_2-1)e_2^2 e_2'}{2!} + \frac{\alpha_2^2(\alpha_2-1)e_2 e_2'^2}{2!} - \frac{\alpha_2^2(\alpha_2-1)^2 e_2^2 e_2'^2}{2!} \right]$$

Substitute the expected value of errors term and the value of  $\delta_{\beta x}$  and  $\delta_{\beta z}$  to (15)

$$\text{Bias}(k_1) = \bar{Y} \left\{ \begin{array}{l} \omega_0 \\ + \omega_1 [5 + \alpha_{12}(f_4 R_{12x} + f_3 R_{22x} - f_2 R_{32x})] \\ + \omega_2 [5 + \alpha_{22}(f_4 R_{12z} + f_3 R_{22z} - f_2 R_{32z})] \end{array} \right\} \quad (24)$$

$$\text{where } R_{12x} = (3\alpha_{12} - 2)C_x^2 \delta_{\beta x}^2, R_{22x} = 3\alpha_{12}\rho_{xy}C_x C_y \delta_{\beta x}, R_{32x} = 6\alpha_{11}^2 C_x^2 \delta_{\beta x}^2$$

$$R_{12z} = (3\alpha_{12} - 2)C_z^2 \delta_{\beta z}^2, R_{22z} = 3\alpha_{12}\rho_{zy}C_z C_y \delta_{\beta z}, R_{32z} = 6\alpha_{12}^2 C_z^2 \delta_{\beta z}^2$$

Substitute the expected value of errors term as it's in and  $\delta_{\beta x}$  and  $\delta_{\beta z}$  to the above equation (24)

The Mean Square Error (MSE):

$$\text{MSE}(k_1) = \bar{Y}^2 \left\{ \begin{array}{l} \omega_0^2 [1 + f_1 C_y^2] \\ + \omega_1^2 [25 + f_1 25 C_y^2 + \psi_{12x} f_4 - \psi_{22x} f_3 - \psi_{32x} f_2] \\ + \omega_2^2 [25 + f_1 25 C_y^2 + \psi_{12z} f_4 - \psi_{22z} f_3 - \psi_{32z} f_2] \end{array} \right\} \quad (25)$$

$$\text{where } \psi_{12x} = \alpha_{12}(39\alpha_{12} - 20)\delta_{\beta x}^2 C_x^2, \psi_{22x} = 60\alpha_{12}\rho_{xy}C_x C_y \delta_{\beta x}, \psi_{32x} = 78\alpha_{12}^2 \delta_{\beta x}^2 C_x^2$$

$$\psi_{12z} = \alpha_{21}(39\alpha_{22} - 20)\delta_{\beta z}^2 C_z^2, \psi_{22z} = 60\alpha_{22}\rho_{zy}C_z C_y \delta_{\beta z}, \psi_{32z} = 78\alpha_{22}^2 \delta_{\beta z}^2 C_z^2$$

$$\text{MSE}(k) = \left\{ \bar{Y}^2 \left[ \omega_0^2 \tau_y + \omega_1^2 R_{2x} + \omega_2^2 R_{2z} \right] \right\} \quad (26)$$

$$\text{Where; } \tau_y = [1 + f_1 C_y^2], R_{2x} = [25 + f_1 25 C_y^2 + \psi_{12x} f_4 - \psi_{22x} f_3 - \psi_{32x} f_2],$$

$$R_{2z} = [25 + f_1 25 C_y^2 + \psi_{12z} f_4 - \psi_{22z} f_3 - \psi_{32z} f_2]$$

Optimum value of  $\alpha_{11}$  and  $\alpha_{21}$  for Estimator  $k_1$ .

$$\alpha_{11} = \frac{20\delta_{\beta x}^2 C_x^2 f_4 - 60\rho_{xy}C_x C_y \delta_{\beta x} f_3}{78\delta_{\beta x}^2 C_x^2 f_4 - 156\delta_{\beta x}^2 C_x^2 f_2} = \frac{10\delta_{\beta x} f_4 - 3K_{xy} f_3}{[39f_4 - 78f_2]\delta_{\beta x}}, \alpha_{21} = \frac{20\delta_{\beta z}^2 C_z^2 f_4 - 60\rho_{zy}C_z C_y \delta_{\beta z} f_3}{78\delta_{\beta z}^2 C_z^2 f_4 - 156\delta_{\beta z}^2 C_z^2 f_2} = \frac{10\delta_{\beta z} f_4 - 3K_{zy} f_3}{[39f_4 - 78f_2]\delta_{\beta z}}$$

### 6. Data Analysis and Discussion of result

The data used in this research work was the measurements of weights in kilogram (kg), heights in centimeter (cm) and ages in (year) of Nigerian student's collected at School of Pure and Applied Sciences, Federal Polytechnic Ado-Ekiti Ekiti State.

The weights stand for the study variable  $\bar{y}$ , while the heights and the ages stand for the first and second auxiliary  $\bar{x}$  and  $\bar{z}$  variables respectfully. The first sample ( $n'$ ) 800 students selected from population ( $N$ ) of 1010 students, the two auxiliary variables  $\bar{x}'$  and  $\bar{z}'$  were measured and the second sample ( $n$ ) 500 students selected from the first sample( $n'$ ), where the study variable ( $\bar{y}$ ) and the two auxiliary variables  $\bar{x}$  and  $\bar{z}$  were measured respectively. The following are the measurements values obtained.

$$N=1010, n'=800, n=500, \bar{x}' = 19.9, \bar{z}' =160.21, \bar{y} =6.962, \bar{x} = 22.112, \bar{z} =164.98, C_y = 21.692, C_x= 8.9212, C_z=5.302, \rho_{yx} = -0.007, \rho_{yz} =0.049, \beta_{yx} = -0.047, \beta_{yz} =0.0753,$$

The analysis in this study was categorized into two. Firstly, the analysis of the efficiency of the modified estimators  $k_1$  and  $k_2$  over the existing estimators  $t_i$  with only the condition of weights alone. i.e when the sum of the weight is equal to one ( $\sum \omega_i=1$ ), i.e when  $\omega_0 < \omega_1 < \omega_2$ ,  $\omega_1 < \omega_0 < \omega_2$  and  $\omega_2 < \omega_0 < \omega_1$  for the condition mentioned above, where;  $\omega_0$ ,  $\omega_1$  and  $\omega_2$  are the weight of study and auxiliary variables  $y$ , and  $x, z$  respectively..

The descriptions of the conditions and the cases weight are stated below appropriately. The weight of the study variable ( $y$ ) and the weight of the auxiliary variable ( $x$  and  $z$ ) signified ( $\omega_0$ ,  $\omega_1$  and  $\omega_2$ ) respectively. And the weights ( $\omega_i$ ;  $i = 0, 1, 2$ ) were assumed to be one decimal value, where  $\omega_i \in (0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$

Table 1: The Biases and Mean square errors of the existing and the new estimators ( $t_i$  and  $k_1$ ).when  $\sum \omega_i=1$  and ( $\omega_0 < \omega_1 < \omega_2$ )

Estimators	Weight values			Biases	MSEs
	$\omega_0$	$\omega_1$	$\omega_2$		
Cochran (1940) $t_1$				33.17	1151.21
Singh and Vishwakarma (2007) $t_8$				33.39	1161.45
Vishwakarma and kumar (2014), $t_{11}$				32.15	1147.31
Janbandhu, et .al (2023) $t_{14}$				28.93	1124.31
Miriel (2024), $t_{15}$				26.65	1112.60
Kurbah and Khongj (2024), $t_{16}$				24.32	946.43
Bhatt. et al(2025), $t_{17}$				24.30	934.64
Singh and Nigam (2022), $t_{13}$				12.45	179.5
Modified estimator $k_1$	0.1	0.2	0.7	<b>-1.02</b>	<b>2.21</b>
Singh and Nigam (2022), $t_6$				-8.41	189.1
Modified estimator $k_1$	0.1	0.3	0.6	<b>1.13</b>	<b>2.23</b>
Singh and Nigam (2022), $t_6$				9.23	194.2
Modified estimator $k_1$	0.1	0.4	0.5	<b>1.32</b>	<b>2.24</b>
Singh and Nigam (2022), $t_6$				8.13	100.1
Modified estimator $k_1$	0.2	0.3	0.5	<b>-1.46</b>	<b>2.20</b>

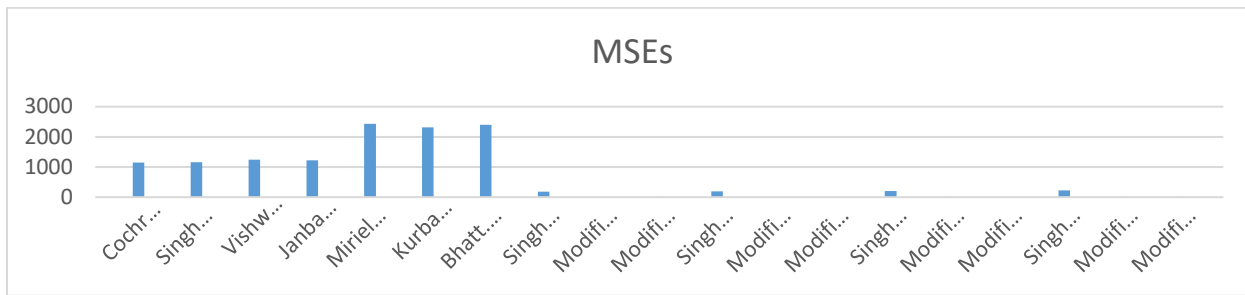


Figure 1: The plot of MSE of the existing and the modified estimators ( $t_i$  and  $k_1$ ) when  $\sum \omega_i = 1$  and  $(\omega_0 < \omega_1 < \omega_2)$

Table 2: The Biases and Mean square errors of the existing and the new estimators ( $t_i$  and  $k_1$ ) when  $\sum \omega_i = 1$  and  $(\omega_1 < \omega_0 < \omega_2)$

Estimators	Weight values			Biases	MSEs
	$\omega_0$	$\omega_1$	$\omega_2$		
Cochran (1940) $t_1$				33.17	1151.21
Singh and Vishwakarma (2007) $t_8$				33.39	1161.45
Vishwakarma and kumar (2014), $t_{11}$				32.15	1147.31
Janbandhu, et .al (2023) $t_{14}$				28.93	1124.31
Miriel (2024), $t_{15}$				26.65	1112.60
Kurbah and Khongj (2024), $t_{16}$				24.32	946.43
Bhatt. et al(2025), $t_{17}$				24.30	934.64
Singh and Nigam (2022), $t_{13}$				-13.01	184.1
<b>Modified estimator <math>k_1</math></b>	0.2	0.1	0.7	<b>1.34</b>	<b>2.45</b>
Singh and Nigam (2022), $t_6$				- 13.31	192.1
<b>Modified estimator <math>k_1</math></b>	0.3	0.1	0.6	<b>-1.35</b>	<b>2.68</b>
Singh and Nigam (2022), $t_6$				12.14	201.1
<b>Modified estimator <math>k_1</math></b>	0.3	0.2	0.5	<b>-1.43</b>	<b>2.96</b>

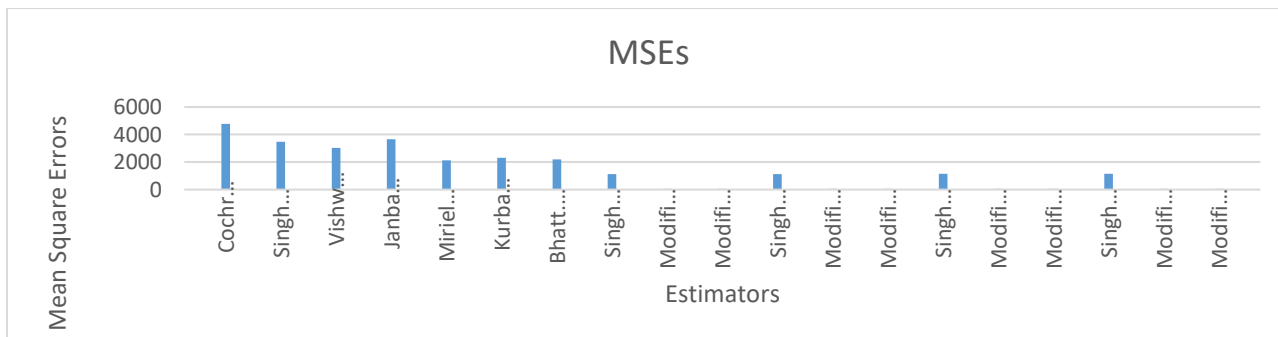


Figure2: The plot of MSE of the existing and the modified estimators ( $t_i$  and  $k_1$ ) when  $\sum \omega_i = 1$  and  $(\omega_1 < \omega_0 < \omega_2)$ .

Table3: The Biases and Mean square errors of the existing and the new estimators ( $t_i$  and  $k_1$ ) when  $\sum \omega_i=1$  and  $(\omega_2 < \omega_0 < \omega_1)$

Estimators	Weight values			Biases	MSEs
	$\omega_0$	$\omega_1$	$\omega_2$		
Cochran (1940) $t_1$				33.17	1151.21
Singh and Vishwakarma (2007) $t_8$				33.39	1161.45
Vishwakarma and kumar (2014), $t_{11}$				32.15	1147.31
Janbandhu, et .al (2023) $t_{14}$				28.93	1124.31
Mirieli (2024), $t_{15}$				26.65	1112.60
Kurbah and Khongj (2024), $t_{16}$				24.32	946.43
Bhatt. et al(2025), $t_{17}$				24.30	934.64
Singh and Nigam (2022), $t_{13}$				-13.13	161.9
Modified estimator $k_1$	0.2	0.7	0.1	-1.27	2.33
Singh and Nigam (2022), $t_6$				12.08	162.56
Modified estimator $k_1$	0.3	0.6	0.1	-1.25	2.41
Singh and Nigam (2022), $t_6$				13.31	165.35
Modified estimator $k_1$	0.3	0.5	0.2	1.38	2.61

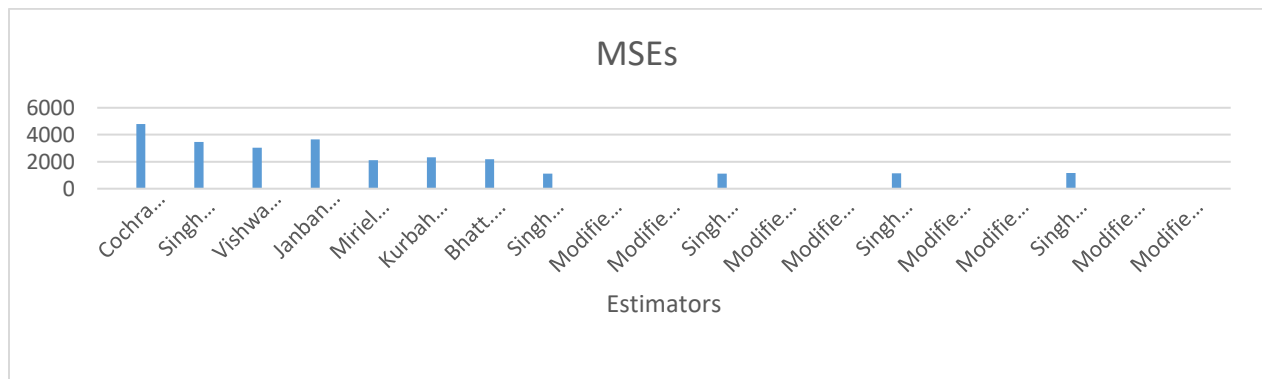


Figure3: The plot of MSE of the existing and the modified estimators ( $t_i$  and  $k_1$ ) when  $\sum \omega_i=1$  and  $(\omega_2 < \omega_0 < \omega_1)$

**Discussion of the Results**

The results obtained from both the analytical derivations and empirical evaluations clearly demonstrate the superior performance of the proposed modified estimator ( $k_1$ ) relative to the existing estimators ( $t_1, t_8, t_{11}, t_{13}, t_{14}, t_{15}$ ) and ( $t_{16}$ ). The comparative assessment was primarily based on the Mean Square Error (MSE), which serves as a comprehensive measure of estimator

efficiency by jointly accounting for variance and bias components.

From Tables 1-3, it is evident that the proposed estimator ( $k_1$ ) consistently attains the minimum MSE across all considered scenarios. This indicates a substantial gain in efficiency over the competing estimators, reflecting improved precision in estimating the finite population mean. The observed reduction in MSE can be attributed to the structural

advantages of the modified estimator, particularly its ability to effectively integrate auxiliary information and stabilize variability through its formulation. In contrast, the existing estimators exhibit relatively higher MSE values, suggesting greater dispersion and lower reliability in their estimates.

The graphical illustrations presented in Figures 1-3 further reinforce these findings. The bar charts provide a visual comparison of the MSE values, where the bars corresponding to the existing estimators are consistently taller than that of the proposed estimator ( $k_1$ ). This graphical trend clearly indicates that the variability associated with the traditional estimators is higher, whereas the modified estimator maintains a comparatively lower error profile. The consistency between the tabular and graphical results strengthens the validity of the conclusions drawn from this study.

From a methodological standpoint, the improved performance of ( $k_1$ ) suggests that the incorporation of auxiliary information in a refined functional form enhances estimator efficiency under the sampling framework considered. The estimator effectively captures additional information embedded in the auxiliary variables, thereby reducing estimation uncertainty.

Overall, the results confirm that the proposed modified estimator ( $k_1$ ) is more efficient than the existing estimators examined in this study. Its ability to achieve lower MSE values across multiple evaluation criteria highlights its robustness and suitability for practical application in survey sampling, particularly in situations where improved precision of population mean estimation is required.

Therefore, the suggested estimators  $k_1$  are better than existing estimators considered in this study. But the modified estimator  $k_1$  efficiently better than the existing estimators whenever  $\omega_0 = 0.2$ ,  $\omega_1 = 0.3$  and  $\omega_2 = 5$ , when  $\sum \omega_i = 1$ , and ( $\omega_0 < \omega_1 < \omega_2$ )

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