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Look-Ahead Linear Multistep Methods for Initial Value Problems with Quadratic Stability Polynomials

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Abstract

This study presents a recently developed class of approximate numerical integration techniques for solving first-order ordinary differential equations arising from initial value problems (IVPs). The proposed family of schemes, referred to as *look-ahead linear multistep methods*, is formulated based on the introduction of a function evaluation at an additional step beyond the current integration point. This look-ahead strategy enhances the computational structure of the methods and improves their numerical performance in the integration of a wide range of ordinary differential equations. One of the major advantages of the proposed approach is its ability to generate relatively large regions of absolute stability, particularly for methods whose stability polynomials possess quadratic characteristic roots. The methods further exhibit predictor-corrector characteristics, thereby improving convergence behavior, numerical consistency, and solution refinement during implementation. However, owing to the combined predictor-corrector formulation and the inherent look-ahead mechanism, the dependence of the stability roots on the independent variable also becomes quadratic in nature. In this work, the coefficients of the developed methods are derived explicitly up to order seven, while a systematic strategy for constructing higher-order schemes is also proposed. Preliminary numerical experiments conducted on selected initial value problems demonstrate that the developed methods are computationally efficient, highly accurate, and numerically reliable. Comparative analyses further reveal that the proposed class of look-ahead multistep methods performs competitively, and in several cases more effectively, than existing methods possessing similar structural and stability properties.

Keywords

Continuous scheme,
Differential system,
Discrete variable,
Initial value problem,
Non-linear system,

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1 Introduction

The following types of linear and nonlinear systems of initial value problems of ordinary differential equations (ODEs) are of interest.

$$\begin{cases} \frac{du(t)}{dt} = f(t, u(t)), & t \in I, \\ u(t_0) = u_0, \end{cases} \tag{1}$$

where the unknown $f(t, u)$ is a given real value function in the strip $s = [a, b] \times (-\infty, +\infty)$, which is continuous with respect to both variables (t, u) and the initial vector u_0 in $C^1[a, b]$. It is assumed that f satisfies some conditions that guarantee the existences and uniqueness of a solution $u(t) \in C^1[a, b]$ as an initial value problem (IVPs). Among many numerical integration methods, the discrete variable methods (DVMs)[1] with a constant step-size h have the capability of generating the approximate solutions u_n on the step-point $t_n = a + nh$. Many existing methods such as the Euler method, Runge-Kutta methods and the linear multistep methods fall into the category of discrete variable methods. Due to its versatility in solving generic initial value problems of ODEs, the methods (DVMs) have been extensively used for IVPs. We note that in order to deal with a system of ODEs directly, a discrete variable approach should be "linear" with regard to the functional values u and f to cope with system of initial value problem in ordinary differential equations. Secondly, the step size h is assumed to be constant, in particular for theoretical analysis and an adaptive step size control is significant for practice [2]. Note, that numerical analysts are still pursuing a new method with better performance and higher reliability for the

numerical solutions of initial value problems of ordinary differential equations.

In view of the above, the present note is devoted to propose a variant of linear multistep method (LMM) in a "look-ahead" manner. We give remark of some of the features of the look-ahead linear multistep methods as follows: let k be the number of steps incorporated in a method (a k -step method), we assume that the back values $u_n, u_{n+1}, u_{n+2}, \dots, u_{n+k-1}$ are available, with the present step point u_{n+k-1} , we compute u_{n+k} (the look for value) in a systematic way by the information u_{n+j} and $f(t_{n+j}, u_{n+j})$,

($j = 0, 1, \dots, k - 1$). Therefore, in this note we show some historical background and concept of "look-ahead" linear multistep methods (LALMMs), though they were not called "look-ahead" linear multistep methods by then but were efficient with good stability properties for the resolution of equation (1). One example of this class is the third-order convergence as well as A -stability discrete variable method (DVM) proposed by Usamani and Agarwal [3] which is a typical look-ahead linear multistep method. Their method states that, from the step point t with an approximate solution u_0 for $u(t)$, and further assume an already calculated approximate solution $u_1^{[0]} = u_1$ for $u(t+h)$. Then, for $m = 0, 1, \dots$, repeat the predictor-corrector iteration by

$$\begin{cases} u_2^{[m]} = 5u_0 - 4u_1^{[m]} + 2h \{ f(t_0, u_0) + 2f(t_1, u_1^{[m]}) \} \\ u_1^{[m+1]} = u_0 + \frac{h}{12} \{ 5f(t_0, u_0) + 8f(t_1, u_1^{[m]}) - f(t_2, u_2^{[m]}) \}. \end{cases}$$

The predictor-corrector procedure is repeated until the correction's size is suitably modest, or until the estimate

$$\|u_{n+k}^{[m+1]} - u_{n+k}^{[m]}\| \leq \delta_{TOL}, \quad (2)$$

is acquired for a predetermined δ_{TOL} . The converged value $u_1^{[m+1]}$ is taken as for the corrected approximation of $u(t+h)$, and start the Predictor-Corrector (PC) iteration on the

right-shifted step-point $t_{n+1}, t_{n+2}, \dots, t_{n+k+1}$ with $u_1^{[m+1]}$ and $u_2^{[m]}$ playing the role of u_0 and $u_1^{[0]}$ respectively on the previously calculated step point. That is, the following stepping diagram is employed

$$\begin{aligned} & (t_0, u_0), (t_1, u_1^{[m+1]}), (t_2, u_2^{[m]}) \\ & \rightarrow (t_1, u_1), (t_2, u_2^{[0]}), (t_3, u_3), \end{aligned}$$

see, for example Figure (1a), for the mechanism of LALMMs.

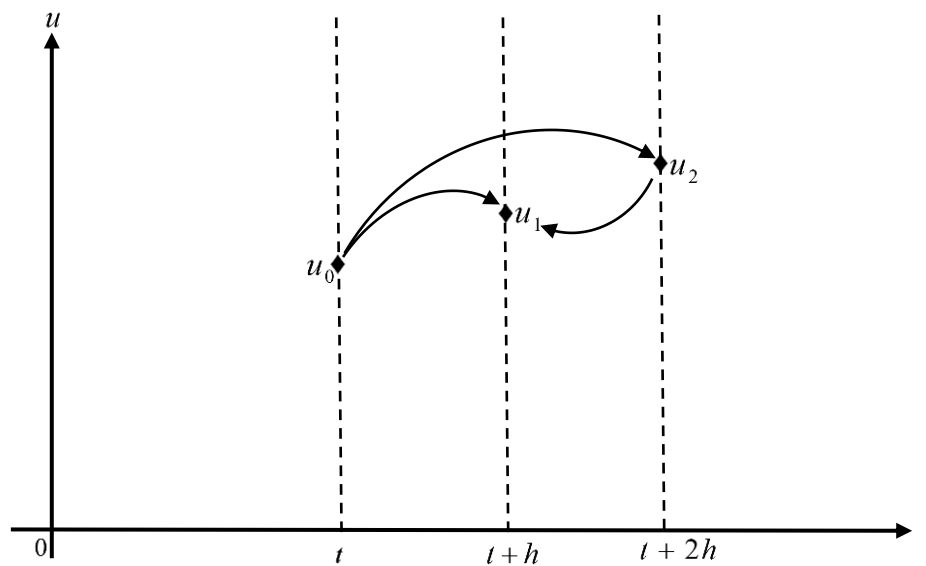


Figure 1a: Mechanism of Look-Ahead Linear Multistep Method

Usmani and Agarwal demonstrated the third-order convergence of their method as well as A -stability suitable for initial value problems.

Following the result, Jacques [4] also derived a new pair of Predictor-Corrector (PC) which attains L -stable keeping the third-order convergence written in the form,

$$\begin{cases} u_2^{[m]} = u_0 + 2hf(t_1, u_1^{[m]}), \\ u_1^{[m+1]} = u_0 + \frac{h}{12} \{5f(t_0, u_0) + 8f(t_1, u_1^{[m]}) - f(t_2, u_2^{[m]})\}, \quad \text{for } m = 0, 1, \dots \end{cases} \quad (3)$$

This scheme attains L -stability, keeping the third-order convergence. Note that in spite of the fact that the predictor equation is nothing but the midpoint rule a good stability property is obtained for the method. A survey can easily revealed that the LALMM idea can be traced back to other earlier works. Some of these are the extended-backward differentiation formula (EBDF) methods of Cash

[5, 6]. Particularly in [6] a k -step implicit DVM, employing the back values $u_n, u_{n+1}, \dots, u_{n+k-1}$ of the form

$$\sum_{j=0}^k \alpha_j u_{n+j} = h \beta_k f_{n+k} \tag{4}$$

where $f_j = f(t_j, u_j)$ was derived. The formula parameters α_j and β_k satisfying the order conditions

$$\sum_{j=0}^k \alpha_j j^q = q \beta_k k^{q-1}, \text{ for } q = 0, 1, \dots, k,$$

gives the k -th order method. Its feature lies on the fact that BDFs up to $k = 6$ attain the stiff stability. Of course, BDF of $k = 1$ is nothing but the implicit Euler scheme, and is A -stable, whereas others cannot be A -stable. For $k = 6$ the BDF is written as

$$\frac{147}{60} u_{n+6} - 6 u_{n+5} + \frac{15}{2} u_{n+4} - \frac{20}{3} u_{n+3} + \frac{15}{4} u_{n+2} - \frac{6}{5} u_{n+1} + \frac{1}{6} u_n = h f_{n+6}.$$

An extended BDF or modified BDF by adding a "super-future" points at t_{n+k+1} so that it is described as a modified backward differentiation formula of the form:

$$\sum_{j=0}^k \alpha_j^* u_{n+j} = h \beta_k^* f_{n+k} + h \beta_{k+1}^* f_{n+k+1} \tag{5}$$

The parameters of the formula are determined to satisfy the order conditions

$$\sum_{j=0}^k \alpha_j^* j^q = q \sum_{j=k}^{k+1} \beta_j^* j^{q-1} \text{ for } q = 0, 1, \dots, k+1. \tag{6}$$

Then it attains $(k+1)$ -st order of convergence. Usually the normalization $\alpha_k^* = 1$ is also imposed.

Inserting (5) into (6) Cash established the EBDF algorithm as follows:

1. From $u_n, u_{n+1}, \dots, u_{n+k-1}$, compute \bar{u}_{n+k} as the solution of the conventional BDF formula.
2. Compute \bar{u}_{n+k+1} as the solution of the same BDF formula advanced by one step,

$$\sum_{j=0}^{k-1} \alpha_j u_{n+k+1} + \alpha_k \bar{u}_{n+k+1} = h \beta_k f(t_{n+k+1}, u_{n+k+1}), \text{ where } u_{n+k} \text{ is substituted by } \bar{u}_{n+k}.$$

3. For u_{n+k} , solve the EBDF

$$\sum_{j=0}^k \alpha_j^* u_{n+j} = h \beta_k^* f(t_{n+k}, u_{n+k}) + h \beta_{k+1}^* f(t_{n+k+1}, \bar{u}_{n+k+1}).$$

From the above description, regarding t_{n+k-1} as the current step-point, we can interpret u_{n+k} and u_{n+k+1} as the "look for" value and the "look-ahead" value respectively. Thus, we count Cash's algorithm among the LALMMs. Note that, different from the preceding algorithm, Cash's one does not require iterations till the local convergence.

We now describe the general framework of the LALMMs. Assume that the back values $u_n, u_{n+1}, \dots, u_{n+k-1}$ and the initial guess $u_{n+k}^{[0]}$ are pre-assigned. A LALMM pair is given by

$$\text{Predictor: } u_{n+k+1}^{[\ell]} + \alpha_k u_{n+k}^{[\ell]} + \sum_{i=0}^{k-1} \alpha_i u_{n+i} = h \left(\beta_k f(t_{n+k}, u_{n+k}^{[\ell]}) + \sum_{i=0}^{k-1} \beta_i f(t_{n+i}, u_{n+i}) \right), \tag{7}$$

$$\text{Corrector: } u_{n+k}^{[\ell+1]} + \sum_{i=0}^{k-1} \alpha_i^* u_{n+i} = h \left(\beta_{k+1}^* f(t_{n+k+1}, u_{n+k+1}^{[\ell]}) + \beta_k^* f(t_{n+k}, u_{n+k}^{[\ell]}) + \sum_{i=0}^{k-1} \beta_i^* f(t_{n+i}, u_{n+i}) \right), \quad (8)$$

for $\ell = 0, 1, 2, \dots$. Note that the left-hand side of the corrector equation does not include u_{n+k+1} . We repeat the iteration until (local) convergence is achieved as in (2). Note that $\alpha_j \neq \alpha_j^*$, $\beta_j \neq \beta_j^*$.

Associated with the above, we define the modified characteristic polynomials as follows: -

$$\begin{cases} \rho(\zeta) = \sum_{i=0}^k \alpha_i \zeta^i, & \sigma(\zeta) = \sum_{i=0}^k \beta_i \zeta^i \\ \rho^*(\zeta) = \zeta^k + \sum_{i=0}^{k-1} \alpha_i^* \zeta^i, & \sigma^*(\zeta) = \sum_{i=0}^k \beta_i^* \zeta^i \end{cases} \quad (9)$$

An introduction of the shift operator S with the reference h simply expresses the predictor and corrector equations as

$$\text{Predictor: } \{S^{k+1} + \rho(S)\}u_n = h\sigma(S)f_n \quad (10a)$$

$$\text{Corrector: } \rho^*(S)u_n = h\{\beta_{k+1}^* S^{k+1} + \sigma^*(S)\}f_n \quad (10b)$$

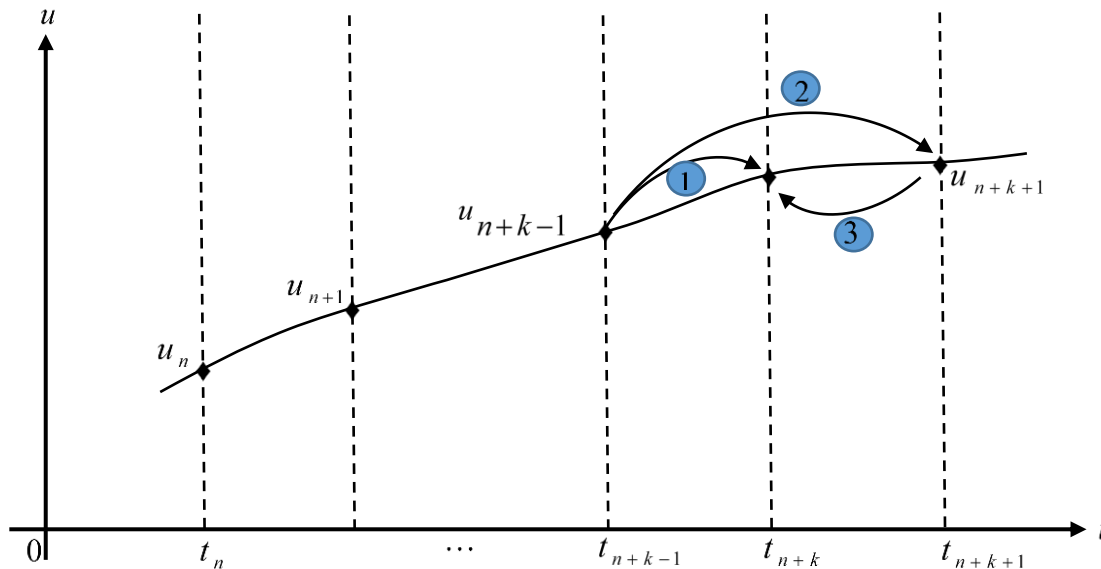


Figure 1b: Schematic Diagram of LALMM Mechanism with Second Derivative

Further, we describe look-ahead linear multistep method with second derivative evaluation as follows: From Figure 1b, we suppose a certain approximation of the "look-for" value $u_{n+k}^{[0]}$ is available. Then we compute the "look-ahead" value u_{n+k+1} by a linear multistep scheme employing $u_n, u_{n+1}, \dots, u_{n+k}^{[0]}$ and their derivatives. Another linear multistep scheme incorporating u_n to u_{n+k+1} corrects the "look-for" value as $u_{n+k}^{[1]}$. From the Figure, regarding t_{n+k-1} as the current step-point, we can interpret u_{n+k} and u_{n+k+1} as the "look-for" and the "look-ahead" values, respectively.

Historically, Urabe [7] proposed a new type of DVM, which falls into the class of LALMMs with second derivative evaluation. One of the features of Urabe's method is the introduction of the second derivative of $u(t)$ into equation (1), that is,

$$g(t, u) = \frac{\partial f}{\partial t}(t, u) + \frac{\partial^2 f}{\partial t^2} f(t, u)$$

into the numerical scheme. Urabe's method takes the predictor-corrector form like the Usmani and Agarwal method. Starting from the step-point t with an approximate solution u_0 of $u(t)$ we again assume an already-calculated approximate solution $u_1^{[0]} = u_1$ for $u(t+h)$. Then for $m = 0, 1, \dots$, repeat the predictor-corrector iteration,

$$\begin{cases} u_2^{[m]} = -31u_0 + 32u_1^{[m]} - h\{14f(t_0, u_0) + 16f(t_1, u_1^{[m]})\} + h^2\{-2g(t_0, u_0) + 4g(t_1, u_1^{[m]})\}, \\ u_1^{[m+1]} = u_0 + \frac{h}{240}\{101f(t_0, u_0) + 128f(t_1, u_1^{[m]}) + 11f(t_2, u_2^{[m]})\} + \\ \frac{h^2}{240}\{13g(t_0, u_0) - 40g(t_1, u_1^{[m]}) - 3g(t_2, u_2^{[m]})\} \end{cases} \quad (11)$$

until the local convergence is achieved. The predictor equation is of order 6, while the corrector is of order 5. Urabe applied the scheme successfully to the computations of nonlinear oscillation analysis. Therefore, the focus of the study is to develop theoretical and computational aspects of LALMM considering the historical background into account.

2. Procedure for the Derivation of the Look-Ahead Linear Multistep Methods

This section describes the process for deriving the Look-Ahead Linear Multistep Methods for the continuous numerical integration of first-order ordinary differential equations. Here, we use the interpolant of the type in (12) below to assume an approximate solution to the exact solution of equation (1):

$$u(t) = \alpha_0 + \alpha_1 t + \dots + \alpha_{p-1} t^{p-1} = \sum_{i=0}^{p-1} \alpha_i t^i, \quad (12)$$

which is continuously differentiable of any order required. We set the sum $P = r + s + v$ so as to be able to determine $\{\alpha_i\}$ in (12). In this formulation r denotes the number of interpolation points used and, $s > 0, v > 0$ are distinct collocation points. Interpolating $u(t)$ in (12) at the points $\{t_{n+j}\}$ and collocating $u'(t)$ and $u''(t)$ at the same points $\{t_{n+j}\}$ we have the following system of equations

$$u(t_{n+j}) = u_{n+j}, \quad (j = 0, 1, \dots, r-1), \quad (13a)$$

$$u'(t_{n+j}) = f_{n+j}, \quad (j = 0, 1, \dots, s-1), \quad (13b)$$

$$u''(t_{n+j}) = g_{n+j}, \quad (j = 0, 1, \dots, v-1). \quad (13c)$$

In the sense of Mitsui and Yakubu [8, 9] the system of equations (13(a-c)) can be expressed in the matrix-vector form as

$$D\alpha = u, \quad (14)$$

where the p -square matrix D is non-singular, the p -vectors α , and u are defined as follows:

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{p-1})^T, u = (u_n, \dots, u_{n+r-1}, f_n, \dots, f_{n+s-1}, g_n, \dots, g_{n+v-1})^T. \quad (15)$$

Similar to the Vandermonde matrix, D in (14) is invertible since we assumed D to be non-singular. By taking into account the inverse of the Vandermonde matrix like D , a closed version of the solution for the system in (14) is obtained, that is,

$$\alpha = Cu \quad (C = D^{-1}). \quad (16)$$

Rearranging equations (13-16) to obtain the multistep collocation formula with second derivative evaluation as follows:

$$u(t) = \sum_{j=0}^{r-1} \alpha_j(t) u_{n+j} + h \sum_{j=0}^{s-1} \beta_j(t) f_{n+j} + h^2 \sum_{j=0}^{v-1} \gamma_j(t) g_{n+j}. \quad (17)$$

where

$$\begin{aligned} u_{n+j} &= u(t_n + jh), \\ f_{n+j} &= f(t_n + jh, u(t_n + jh)), \\ g_{n+j} &= f_t + fu' = f_t + ff_u. \end{aligned}$$

The parameters $\alpha_j(t)$, $\beta_j(t)$, and $\gamma_j(t)$ are continuous coefficients of the formula which are to be determined. They are polynomials of degree $p-1$ given as

$$\alpha_j(t) = \sum_{i=0}^{p-1} \alpha_{j,i+1} t^i, \quad h\beta_j(t) = \sum_{i=0}^{p-1} h\beta_{j,i+1} t^i, \quad h^2\gamma_j(t) = \sum_{i=0}^{p-1} h^2\gamma_{j,i+1} t^i.$$

The numerical constant coefficients $\alpha_{j,i+1}$, $\beta_{j,i+1}$, and $\gamma_{j,i+1}$ are to be determined. They are obtained from the components of the matrix D^{-1} . That is, if the identity

$$C = D^{-1} \quad (18)$$

holds good. The choice $C = D^{-1}$ leads to the determination of the numerical constant coefficients $\alpha_{j,i+1}$, $\beta_{j,i+1}$ and $\gamma_{j,i+1}$. Actual evaluation of the matrices C and D are carried out with a computer algebra system, for example, Maple to determine the constant coefficients. In the second derivative methods, we see that not only the function $f(t, u)$ is evaluated at the internal intermediate points, but in addition the functions Df where D is the differential operator are evaluated at the internal intermediate points. Hence, in addition to the computation of the f -values at the internal stages in the standard linear multistep methods, the second derivative "look-ahead" linear multistep methods involve computing g -values where f and g are as defined in (17).

3. Specification of the Methods

3.1 "Look-Ahead" Linear Multistep Method of Order Five

In this section, we use the multistep collocation formula (17) for the construction of the first "look-ahead" linear multistep method. We construct a propose continuous scheme of the corrector formula of order five as,

$$u(t) = \alpha_0(t)u_n + h \sum_{j=0}^2 \beta_j(t)f_{n+j} + h^2 \sum_{j=0}^2 \gamma_j(t)g_{n+j}. \tag{19a}$$

Expanding (19a) fully we obtain the continuous scheme of the corrector formula in the form

$$u(t) = \alpha_0(t)u_n + h[\beta_0(t)f_n + \beta_1(t)f_{n+1} + \beta_2(t)f_{n+2}] + h^2[\gamma_0(t)g_n + \gamma_1(t)g_{n+1} + \gamma_2(t)g_{n+2}]. \tag{19b}$$

Evaluating the proposed continuous scheme (19b) at the point $t = t_{n+1}$ we obtain the first corrector formula of the "look-ahead" linear multistep method with second derivative evaluation as

$$u_{n+1} = u_n + \frac{h}{120}[55f_n + 64f_{n+1} + f_{n+2}] + \frac{h^2}{120}[8g_n - 14g_{n+1}] \tag{20}$$

and the predictor formula as given byUrabe [7] as follows

$$u_{n+2} = -3lu_n + 32u_{n+1} + h[-14f_n - 16f_{n+1}] + h^2[-2g_n + 4g_{n+1}].$$

3.2 Look-Ahead Linear Multistep Method of Order Seven

For the higher order "look-ahead" linear multistep method of order seven we have the following interpolation and collocation polynomial for obtaining the propose continuous scheme of the corrector formula as follows:

$$u(t) = \alpha_0(t)u_n + h \sum_{j=0}^3 \beta_j(t)f_{n+j} + h^2 \sum_{j=0}^3 \gamma_j(t)g_{n+j}. \tag{21a}$$

Similarly, on expansion of (21a) fully using Taylor series method of expansion, we obtain the continuous scheme of the corrector formula as

$$u(t) = \alpha_0(t)u_n + h[\beta_0(t)f_n + \beta_1(t)f_{n+1} + \beta_2(t)f_{n+2} + \beta_3(t)f_{n+3}] + h^2[\gamma_0(t)g_n + \gamma_1(t)g_{n+1} + \gamma_2(t)g_{n+2} + \gamma_3(t)g_{n+3}]. \tag{21b}$$

Evaluating the continuous scheme in (21b) as in the above method of section (3.1) at the point $t = t_{n+2}$ we obtain the corrector formula of the second "look-ahead" linear multistep method of higher order with second derivative evaluation as,

$$u_{n+2} = u_n + \frac{h}{2835}[1324f_n + 3213f_{n+1} + 1242f_{n+2} - 109f_{n+3}] + \frac{h^2}{2835}[186g_n + 81g_{n+1} + 33g_{n+3}] \tag{22}$$

and the predictor formula is given as

$$u_{n+3} = -\frac{79}{2}u_n - 27u_{n+1} + \frac{135}{2}u_{n+2} + \frac{h}{4}[-69f_n - 216f_{n+1} - 135f_{n+2}] + \frac{h^2}{4}[-9g_n + 27g_{n+2}].$$

Remark 1: By evaluating the continuous scheme in (21b) at the point $t = t_{n+1}$, it is simple to get the starting value u_1 for the second method in (22) from the continuous scheme of the corrector formula for computing purposes. The computational efficiency will be precisely attained without compromising the method's accuracy since the starting value u_1 is obtained from the same continuous scheme as the corrector formula.

4. Analysis of the properties of the LALM numerical integrators

4.1 The order of the look-ahead linear multi-step numerical integrators

We now give the definition of the order of accuracy of the new methods.

Definition 4.1: Mitsui [10] *Predictor is accurate of order p if and only if the estimation*

$$\{S^{k+1} + \rho(S)\}u(t) - h\sigma(S)u'(t) = Ch^{p+1} + O(h^{p+2})$$

holds for any sufficiently smooth solution $u(t)$.

Definition 4.2: Mitsui [10] *Corrector is accurate of order q if and only if the estimation*

$$\rho^*(S)u(t) - h\{\beta_{k+1}^*S^{k+1} + \sigma^*(S)\}u'(t) = \tilde{C}h^{q+1} + O(h^{q+2})$$

holds for any sufficiently smooth solution $u(t)$. Here, the constants C and \tilde{C} may depend on the IVP and t but does not on h .

We examine the characteristics of the recently developed techniques in this section. In order to achieve this, we thoroughly analyze the stability theory of the new LALMMs by examining their characteristics properties, which include stability analysis, order and consistency, zero stability, and order of convergence. Using the second derivative approach in (22) above, we show how to determine the order of the method following Lambert [11].

$$\alpha_0 = -\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \alpha_1 = \begin{bmatrix} 0 \\ 1 \\ -\frac{79}{2} \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 \\ 0 \\ -\frac{54}{2} \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{135}{2} \end{bmatrix},$$

$$\beta_0 = \begin{bmatrix} 0 \\ \frac{1324}{2835} \\ -\frac{69}{4} \end{bmatrix}, \beta_1 = \begin{bmatrix} 0 \\ \frac{3213}{2835} \\ -\frac{216}{4} \end{bmatrix}, \beta_2 = \begin{bmatrix} 0 \\ \frac{1242}{2835} \\ \frac{135}{4} \end{bmatrix}, \beta_3 = \begin{bmatrix} 0 \\ -\frac{109}{2835} \\ 0 \end{bmatrix},$$

Definition 4.3: Lambert [11] (*Consistency*) If the methods' sequence satisfy the inequality, $p \geq 1$, then the family of methods in (17) are consistent.

- (i) $\rho(1) = 0$,
- (ii) $\rho'(1) = \sigma(1)$, with $\rho(z)$ and $\sigma(z)$ represent the first and second characteristic polynomials, which are independent of each other.

From Table 1 and definition 4.3 we can attest that the "look-ahead" linear multistep methods with second derivatives are consistent.

Definition 4.4: Lambert [11, 13] (*Zero-stability*) It is said that a family of methods of the type in (17) are zero-stable if the roots condition

$$\rho(\lambda) = \det \left[\sum_{i=0}^k A^i \lambda^{k-i} \right] = 0,$$

makes the inequality $|\lambda_j| \leq 1$, $j = 1, 2, 3, \dots, k$. However, the multiplicity shouldn't be higher than two when the inequality becomes $|\lambda_j| = 1$.

By definition 4.4, the newly derived LALMMs are zero-stable.

4.2 Convergence Analysis of LALMMs

Convergence analysis is necessary for LALMMs. Let $u(t)$ be the solution of the initial value problem of the differential equation given in (1). Here $f : [t_0, b] \times R^d \rightarrow R^d$ and $u : [t_0, b] \rightarrow R^d$, with the assumption that the function f is continuously differentiable and its Lipschitz constant is L .

Now the expansion in power series for the predictor equation (7) is as follows:

$$\begin{aligned} u(t + (k+1)h) + \alpha_k u(t + kh) + \sum_{i=0}^{k-1} \alpha_i u(t + ih) - h \sum_{i=0}^k \beta_i u(t + ih) \\ = u(t) + (k+1)hu'(t) + \alpha_k(u(t) + khu'(t)) + \sum_{i=0}^{k-1} \alpha_i(u(t) + iju'(t)) - h \sum_{i=0}^k \beta_i u'(t) + O(h^2) \\ = \left(1 + \alpha_k + \sum_{i=0}^{k-1} \alpha_i \right) u(t) + h \left\{ (k+1) + k\alpha_k + \sum_{i=0}^{k-1} i\alpha_i - \sum_{i=0}^k \beta_k \right\} u'(t) + O(h^2). \end{aligned}$$

This leads to the condition that the predictor is of first order (that is consistent) if

$$1 + \alpha_k + \sum_{i=0}^{k-1} \alpha_i = 0 \quad \text{and} \quad (k+1) + k\alpha_k + \sum_{i=0}^{k-1} i\alpha_i - \sum_{i=0}^k \beta_k = 0.$$

This is equivalent to

$$1 + \rho(1) = 0 \quad \text{and} \quad k + 1 + \rho'(1) - \sigma(1) = 0.$$

Similarly, the condition that the corrector is consistent is

$$1 + \sum_{i=0}^{k-1} \alpha_i^* = 0 \text{ and } k + \sum_{i=0}^k \alpha_i^* - \beta_{k+1}^* - \sum_{i=0}^k \beta_i^* = 0$$

or equivalently

$$\rho^*(1) = 0 \text{ and } \rho'^*(1) - \beta_{k+1}^* - \sigma^*(1) = 0.$$

On the other hand, for local and global errors, we consider the exact solution of equation (1) to satisfy

$$\{S^{k+1} + \rho(S)\}u(t_n) = h\sigma(S)f(t_n, u(t_n)) + T_N$$

and

$$\rho^*(S)u(t_n) = h\{\beta_{k+1}^*S^{k+1} + \sigma^*(S)\}f(t_n, u(t_n)) + \tilde{T}_n,$$

where T_n and \tilde{T}_n are the local truncation errors.

Further, we use second derivative evaluation to examine the order of convergence of the recently developed "look-ahead" linear multistep approaches. In order to appropriately examine the convergence of the approaches, we provide a few definitions. We use the corrector method described in (22) above as an example.

Definition 4.5: Lambert [11] *A function $f(t, u)$ satisfies the Lipschitz condition on the domain $D \subseteq \mathbb{R}^2$ if there exists a constant $L > 0$ such that,*

$$\|f(t, u) - f(t, u^*)\| \leq L\|u - u^*\|,$$

$$\|g(t, u) - g(t, u^*)\| \leq L\|u - u^*\|,$$

for all $(t, u), (t, u^*) \in D$.

Definition 4.6: Lambert [13] *A linear multistep method (LMM) is said to be convergence, if for all initial value problems subject to the above Lipschitz condition;*

$$\lim_{h \rightarrow 0} u_n = u^*(t_n)$$

for all solutions $\{u_n\}$ of the method.

Assuming that the exact solution of u_{n+2} is u_{n+2}^* which is written precisely as

$$u_{n+2}^* = u_n^* + \frac{1324hf_n}{2835} + \frac{3213hf_{n+1}}{2835} + \frac{1242hf_{n+2}}{2835} - \frac{109hf_{n+3}}{2835} + \frac{186h^2g_n}{2835} + \frac{81h^2g_{n+1}}{2835} + \frac{33h^2g_{n+3}}{2835} + \frac{13}{140}h^8u^{(8)}(\xi_n) + R_9$$

where R_9 is the remainder term. Subtract u_{n+2} from u_{n+2}^* to obtain

$$\begin{aligned}
u_{n+2}^* - u_{n+2} &= u_n^* - u_n + \frac{1324h}{2835} [f(t_n, u_n^*) - f(t_n, u_n)] + \frac{3213h}{2835} [f(t_{n+1}, u_{n+1}^*) - f(t_{n+1}, u_{n+1})] + \\
&+ \frac{1242h}{2835} [f(t_{n+2}, u_{n+2}^*) - f(t_{n+2}, u_{n+2})] - \frac{109h}{2835} [f(t_{n+3}, u_{n+3}^*) - f(t_{n+3}, u_{n+3})] \\
&+ \frac{186h^2}{2835} [g(t_n, u_n^*) - g(t_n, u_n)] + \frac{81h^2}{2835} [g(t_{n+1}, u_{n+1}^*) - g(t_{n+1}, u_{n+1})] \\
&+ \frac{33h^2}{2835} [g(t_{n+3}, u_{n+3}^*) - g(t_{n+3}, u_{n+3})] + \frac{13}{140} h^8 u^{*(8)}(\xi_n) + R_9.
\end{aligned}$$

Letting $d_n = u_n^* - u_n, \dots, d_{n+3} = u_{n+3}^* - u_{n+3}, q_n = g_n^* - g_n, \dots, q_{n+3} = g_{n+3}^* - g_{n+3}$, then we have

$$\begin{aligned}
|d_{n+2}| &\leq \left(1 + \frac{1324h}{2835} L\right) |d_n| + \frac{3213h}{2835} L |d_{n+1}| + \frac{1242h}{2835} L |d_{n+2}| - \frac{109h}{2835} L |d_{n+3}| + \frac{186h^2}{2835} L |q_n| - \\
&+ \frac{81h^2}{2835} L |q_{n+1}| + \frac{33h^2}{2835} L |q_{n+3}| + \frac{13h^8}{140} u^{*(8)}(\xi_n) + R_9.
\end{aligned}$$

This simplifies to

$$\begin{aligned}
\left(1 - \frac{1242h}{2835} L\right) |d_{n+2}| &\leq \left(1 + \frac{1324h}{2835} L\right) |d_n| + \frac{3213h}{2835} L |d_{n+1}| - \frac{109h}{2835} L |d_{n+3}| + \frac{186h^2}{2835} L |q_n| \\
&+ \frac{81h^2}{2835} L |q_{n+1}| + \frac{33h^2}{2835} L |q_{n+3}| + \frac{13h^8}{140} u^{*(8)}(\xi_n) + O(h^9).
\end{aligned}$$

As $h \rightarrow 0$ with the exception of $|d_n|$ every other term on the right hand side tends to zero while on the left hand side we are left with $|d_{n+2}|$ and by definition of convergence, we have

$$\lim_{h \rightarrow 0} u_{n+2} = u_{n+2}^*.$$

The convergence of u_{n+2} is established in agreement with [14, 15]. Therefore, it can be shown that the other members of the block, u_{n+1} and u_{n+3} converge. Hence, we have shown that the new numerical integrator (22) is convergent, with the remainder term R_9 defined as

$$R_{p+1} = C_{p+2} h^{p+2} u^{*(p+2)}(\xi) = O(h^9).$$

Therefore, the highest order of convergence of the new method is 9.

4.3 Stability of the LALMMs with Second Derivative Evaluation

Linear stability is very important aspect in the design of algorithm to solve ordinary differential equations. For this reason, we consider for the "look-ahead" linear multistep methods the test equation

$$\frac{du}{dt} = \lambda u, \quad \lambda \in \mathbb{C} \text{ and } R\lambda < 0, \quad (23)$$

with a fixed positive step size h see, [16]. Since the "look-ahead" pairs contain second derivatives $g(t, u)$, it is natural to suppose that $g(t, u) = \lambda^2 u$. Also assume that the back values $u_n, u_{n+1}, \dots, u_{n+k-1}$ and the initial guess $u_{n+k}^{(0)}$ are pre-assigned. The stability polynomial

$$\pi(\zeta, z) = \rho^*(\zeta) - z\sigma^*(\zeta) + \beta_{k+1}^* z \rho(\zeta) - \beta_{k+1}^* z^2 \sigma(\zeta)$$

is of second degree with respect to ζ which make the stability analysis easier. Using the scalar test equation in (23) the first "look-ahead" linear multistep predictor-corrector pair's stability function is,

$$u_2 = h_0(z)u_0 + h_1(z)u_1, \quad z = \lambda h,$$

for the predicting process and for the correcting process is of the form

$$\begin{aligned} u_1 &= c_0(z)u_0 + c_1(z)u_1 + c_2(z)u_2 \\ &= [c_0(z) + c_2(z)h_0(z)]u_0 + [c_1(z) + c_2(z)h_1(z)]u_1. \end{aligned}$$

For the above linear test equation (23) the rational function is given as

$$p(z) = \frac{c_0(z) + c_2(z)h_0(z)}{1 - c_1(z) - c_2(z)h_1(z)}$$

and is called the stability factor of method (20). The region of absolute stability for the method is defined by

$$\mathfrak{R} = \{z \in \mathbb{C} : \Re z < 0, |p(z)| \leq 1\}.$$

Substituting appropriately the above defined polynomials and simplify we have the stability function as

$$p_3(z) = \frac{120 + 24z - 6z^2 - 2z^3}{2(60 - 46z + 15z^2 - 2z^3)}.$$

We now define the function $F(t, u)$ by

$$F(t, u) = |\text{denominator of } p(t + iu)|^2 - |\text{numerator of } p(t + iu)|^2.$$

Evaluating the function $F(t, u)$ we have the identity

$$\begin{aligned} F(t, u) &= 12u^6 + 4u^4(-45t^2 + 330t - 424) + 4u^2(45t^4 - 660t^3 + 2544t^2 \\ &\quad - 4248t + 4132) + 4t(-3t^5 + 66t^4 - 424t^3 + 1428t^2 - 4132t + 6960) \end{aligned}$$

which is a polynomial of degree 2 in u , that is u^2 and each coefficient of powers of u^2 is again a polynomial of t . According to theorem 2 in Mitsui [12] \mathfrak{R} includes the left half-plane of \mathbb{C} .

Proceeding in a similar fashion, using the scalar test equation in (23) the second "look-ahead" linear multistep method (22) predictor-corrector pair stability function is,

$$u_3 = h_0(z)u_0 + h_1(z)u_1 + h_2(z)u_2, \quad z = \lambda h,$$

for the predicting process and the correcting process is of the form

$$\begin{aligned} u_2 &= d_0(z)u_0 + d_1(z)u_1 + d_2(z)u_2 + d_3(z)u_3 \\ &= [d_0(z) + d_3(z)h_0(z)]u_0 + [d_1(z) + d_3(z)h_1(z)]u_1 + [d_2(z) + d_3(z)h_2(z)]u_2 \end{aligned}$$

For the above linear test equation (23) the rational function is given as

$$Q(z) = \frac{d_0(z) + d_3(z)h_0(z) + d_1(z) + d_3(z)h_1(z)}{1 - d_2(z) - d_3(z)h_2(z)}.$$

Also substituting all the defined polynomials appropriately and evaluating them correctly we have the stability function as,

$$Q_4(z) = \frac{420 + 1746z + 864z^2 - 312z^3 - 11z^4}{420 + 906z - 875z^2 + 274z^3 + 33z^4}.$$

Thus, the region of absolute stability of method (22) is

$$\mathfrak{R} = \{z \in \mathbb{C} : \operatorname{Re} z < 0, |p(z)| \leq 1\}.$$

The function $G(t, u)$ is defined by

$$G(t, u) = |\text{denominator of } Q(t + iu)|^2 - |\text{numerator of } Q(t + iu)|^2$$

and the identity is

$$\begin{aligned} G(t, u) &= 968u^8 + 4u^6(-6776t^2 - 19635t + 15247) + 2u^4(33880t^4 + 196350t^3 \\ &\quad - 457410t^2 + 396170t + 54551) + 4u^2(-6776t^6 - 58905t^5 + 228705t^4 \\ &\quad - 396170t^3 - 163653t^2 + 3085380t + 965238) + 2t(484t^7 + 5610t^6 \\ &\quad - 30494t^5 + 79234t^4 + 54551t^3 - 2056920t^2 - 231676t + 1113840). \end{aligned}$$

Similarly, according to [12], \mathfrak{R} includes the left half-plane of \mathbb{C} . As stated in [17,18], $|p(z)|, |Q(z)| \rightarrow \frac{1}{2}$ as $|z| \rightarrow \infty, \operatorname{Re} z < 0$, showing that the two new "look-ahead" linear multistep methods are both stable, hence suitable for system of equations.

5. Numerical Experiments with the LALMMs

We would like to provide some numerical results to test the procedures described in section 3. The Table of values compares the exact values at a few chosen mesh points as well as the absolute errors (AEs) of the

computed results. MatLab programs were used to perform the computations. The tests were conducted in Matlab with a fixed step length to enable a direct comparison. In the computation, we indicate the exact solutions for each problem solved by Ext and the number of function evaluations by *nfe*.

Example 1: Linear differential equation

$$Du(t) = \lambda(\sin t - u(t)), \quad u(0) = 0.$$

Exact solution is: $u(t) = \frac{\lambda^2}{\lambda^2 + 1} \sin t - \frac{\lambda}{\lambda^2 + 1} \cos t + \frac{\lambda}{\lambda^2 + 1} \exp(-\lambda t)$. We solve the problem using the new LALMMs and the output were compared with the exact values as shown in Table 2, with $\lambda = 100$.

Table 2: Absolute errors in example 1's numerical integration

t	u	Method (20) $ u(t) - u_n(t) $	Method (22) $ u(t) - u_n(t) $
5	u	$7.990644028279428 \times 10^{-11}$	$3.024484954886517 \times 10^{-12}$
50	u	$9.259083357245013 \times 10^{-10}$	$3.657727080052092 \times 10^{-11}$
250	u	$3.530290768976769 \times 10^{-09}$	$1.752149909599340 \times 10^{-10}$
500	u	$4.225601274690147 \times 10^{-09}$	$3.245193512276380 \times 10^{-10}$

Example 2: Nonlinear differential equation with periodic and variation magnified [18]

$$Du(t) = \cos t(u(t)(2 - u(t))), \quad u(0) = 1.$$

Its exact solution is: $u(t) = \frac{2}{1 + \exp(-2 \sin t)}$.

We use the new LALMMs to solve the problem, and the results are compared with the exact values as indicated in Table 3, just as the above in example 1.

Table 3: Absolute errors in example 2's numerical integration

t	u	Method (20) $ u(t) - u_n(t) $	Method (22) $ u(t) - u_n(t) $
5	u	$8.437694987151190 \times 10^{-15}$	$7.549516567451065 \times 10^{-15}$
50	u	$1.021405182655144 \times 10^{-13}$	$9.237055564881302 \times 10^{-14}$
250	u	$5.240252676230739 \times 10^{-13}$	$4.627409566637653 \times 10^{-13}$
500	u	$1.006084104915317 \times 10^{-12}$	$9.370282327836321 \times 10^{-13}$

Example 3: Consider the second order initial value problem in ordinary differential equation from [19]:

$$(2t + 3)u''(t) + 4u'(t), \quad u(0) = 1, \quad u'(0) = 1.$$

This can be transformed into equivalent system of two first-order initial value problems of ordinary differential equation as follows:

$$\begin{aligned} Du_1(t) &= u_2(t), & u_1(0) &= 1, \\ Du_2(t) &= -\frac{4u_2(t)}{2t + 3}, & u_2(0) &= 1, \end{aligned}$$

with exact solution as $u_1(t) = \frac{5}{2} - \frac{9}{(2t + 3/2)}, \quad u_2(t) = \frac{9}{(2t + 3/2)^2}.$

The system has singularity at the point $t = -3/2$. The numerical approximate solution over the range of integration interval $[0, 8]$ were obtained and the absolute errors are presented in Table 4. We remark that based on the numerical results in the Table, the two new methods are convergent.

Table 4: Absolute errors in the numerical integration of example 3

t	u_i	Method (20) $ u(t) - u_n(t) $	Method (22) $ u(t) - u_n(t) $
2	u_1	$6.024070131616099 \times 10^{-13}$	$5.750955267558311 \times 10^{-14}$
	u_2	$1.093347634650854 \times 10^{-12}$	$6.983302824892235 \times 10^{-14}$
50	u_1	$2.951172639598099 \times 10^{-11}$	$2.825517597671023 \times 10^{-12}$
	u_2	$5.356193266692344 \times 10^{-11}$	$3.430700168394196 \times 10^{-12}$
250	u_1	$1.498632329344218 \times 10^{-10}$	$1.435651597603282 \times 10^{-11}$
	u_2	$2.720254022037238 \times 10^{-10}$	$1.743216682115190 \times 10^{-11}$
500	u_1	$3.000542037057130 \times 10^{-10}$	$2.876943128171661 \times 10^{-11}$
	u_2	$5.447496898014492 \times 10^{-10}$	$3.493028088996653 \times 10^{-11}$

Example 4: Consider the system of equation from Usmani & Agarwal [3, 20]

$$\begin{aligned} Du_1(t) &= -15.5u_1(t) + 14.5u_2(t) - 13.5, & u_1(0) &= 3, \\ Du_2(t) &= 14.5u_1(t) - 15.5u_2(t) + 16.5, & u_2(0) &= 2. \end{aligned}$$

for which $u_1(t) = e^{-t} + e^{-30t} + 1, u_2(t) = e^{-t} - e^{-30t} + 2$ is the exact solution. The numerical results over the range of integration interval $[0, 50]$ are presented in Table 5. The numerical results in the Table indicate that the two new methods converge to the exact solution.

Table 5: Absolute errors in the numerical solution of example 4

t	u_i	Method (20) $ u(t) - u_n(t) $	Method (22) $ u(t) - u_n(t) $
5	u_1	$6.898193153697552 \times 10^{-03}$	$6.097510145336882 \times 10^{-03}$
	u_2	$6.865415849451662 \times 10^{-03}$	$5.983924514494365 \times 10^{-03}$
50	u_1	$9.780682106335092 \times 10^{-04}$	$3.433403782164834 \times 10^{-04}$
	u_2	$9.780982482245904 \times 10^{-04}$	$3.433683611033800 \times 10^{-04}$
250	u_1	$1.288791295905867 \times 10^{-11}$	$8.104628079763643 \times 10^{-14}$
	u_2	$1.288791295905867 \times 10^{-11}$	$8.126832540256146 \times 10^{-14}$
500	u_1	$8.881784197001252 \times 10^{-16}$	$8.881784197001252 \times 10^{-16}$
	u_2	$8.881784197001252 \times 10^{-16}$	$8.881784197001252 \times 10^{-16}$

Example 5: This problem is helpful for testing recently developed methods. The system's numerical solution's behavior shows that there are certain challenges within the integration interval, particularly when a method with a limited stability region is employed. This system is a non-stiff problem that has been solved by some authors in the literature, indicating that managing the system with an unstable method is extremely difficult.

$$\begin{aligned} Du_1(t) &= u_3(t) - \cos(t), & u_1(0) &= 1, \\ Du_2(t) &= u_3(t) - \exp(-t), & u_2(0) &= 0, \\ Du_3(t) &= u_1(t) - u_2(t), & u_3(0) &= 2. \end{aligned}$$

The exact solution is

$$\begin{aligned} u_1(t) &= \exp(t), \\ u_2(t) &= \sin(t), \\ u_3(t) &= \exp(t) + \cos(t). \end{aligned}$$

We solve the problem using the newly derived "look-ahead" linear multistep methods and the results obtained are shown in Table 6.

Table 6: Absolute errors in the numerical integration of example 5

t	u_i	Method (20) $ u(t) - u_n(t) $	Method (22) $ u(t) - u_n(t) $
5	u_1	$8.567591081032333 \times 10^{-12}$	$1.374234059881019 \times 10^{-12}$
	u_2	$7.764379440902474 \times 10^{-12}$	$6.186957913153243 \times 10^{-13}$
	u_3	$4.440892098500626 \times 10^{-16}$	$8.881784197001252 \times 10^{-16}$
50	u_1	$1.049567099897786 \times 10^{-10}$	$1.683386763318140 \times 10^{-11}$
	u_2	$9.511557054399811 \times 10^{-11}$	$7.579025724884208 \times 10^{-12}$
	u_3	$3.552713678800501 \times 10^{-15}$	$2.220446049250313 \times 10^{-15}$
250	u_1	$5.334106489840451 \times 10^{-10}$	$8.555289809919486 \times 10^{-11}$
	u_2	$4.833857974268982 \times 10^{-10}$	$3.851387345979104 \times 10^{-11}$
	u_3	$5.817568649035820 \times 10^{-14}$	0
500	u_1	$1.069123900521163 \times 10^{-09}$	$1.714730579749357 \times 10^{-10}$
	u_2	$9.688216021531276 \times 10^{-10}$	$7.718254063215219 \times 10^{-11}$
	u_3	$2.251532293939818 \times 10^{-13}$	$8.881784197001252 \times 10^{-16}$

Example 6: This problem is helpful for testing newly developed methods. The system's complex eigenvalues show that there are some issues with the solutions within the integration interval if a method has limited Region of Absolute Stability (RAS).

$$\begin{aligned}
 Du_1(t) &= u_3(t), & u_1(0) &= 1, \\
 Du_2(t) &= u_4(t), & u_2(0) &= 1, \\
 Du_3(t) &= -\exp(-t)u_2(t), & u_3(0) &= 0, \\
 Du_4(t) &= 2\exp(t)u_3(t), & u_4(0) &= 1.
 \end{aligned}$$

The exact solution is

$$\begin{aligned}
 u_1(t) &= \cos(t), \\
 u_2(t) &= e^t \cos(t), \\
 u_3(t) &= -\sin(t), \\
 u_4(t) &= e^t (\cos(t) - \sin(t)),
 \end{aligned}$$

We solve the problem using the new second derivative "look-ahead" linear multistep methods in the interval $[0, 2]$ and the results obtained are shown in Table 7.

Table 7: Absolute errors in the numerical integration of problem 6

t	u_i	Method (20) $ u(t) - u_n(t) $	Method (22) $ u(t) - u_n(t) $
5	u_1	$4.283795540516167 \times 10^{-12}$	$3.671951631645243 \times 10^{-12}$
	u_2	0	0
	u_3	$5.087006172676818 \times 10^{-12}$	$6.699764219225538 \times 10^{-12}$
	u_4	$8.567591081032333 \times 10^{-12}$	$7.343681218685561 \times 10^{-12}$
50	u_1	$5.247180467904400 \times 10^{-11}$	$4.497868744124389 \times 10^{-11}$
	u_2	$2.664535259100376 \times 10^{-15}$	$3.330669073875470 \times 10^{-15}$
	u_3	$6.232055763642514 \times 10^{-11}$	$8.207515358538489 \times 10^{-11}$
	u_4	$1.049480502501865 \times 10^{-10}$	$8.995781897169763 \times 10^{-11}$
250	u_1	$2.665455633987790 \times 10^{-10}$	$2.285336364593604 \times 10^{-10}$
	u_2	$6.816769371198461 \times 10^{-14}$	$8.593126210598712 \times 10^{-14}$
	u_3	$3.167970261119812 \times 10^{-10}$	$4.171444502754779 \times 10^{-10}$
	u_4	$5.332003727431811 \times 10^{-10}$	$4.570885892007937 \times 10^{-10}$
500	u_1	$5.339182429509037 \times 10^{-10}$	$4.579077117483621 \times 10^{-10}$
	u_2	$2.691180611691380 \times 10^{-13}$	$3.448352714485736 \times 10^{-13}$
	u_3	$6.351338519786781 \times 10^{-10}$	$8.361360583943045 \times 10^{-10}$
	u_4	$1.068273469684300 \times 10^{-09}$	$9.158978020451514 \times 10^{-10}$

Example 7: The nonlinear system of two body problem (Kepler Problem) [18, 21]

$$\begin{aligned}
 Du_1(t) &= u_3(t), & u_1(0) &= 1 - e, \\
 Du_2(t) &= u_4(t), & u_2(0) &= 0, \\
 Du_3(t) &= -\frac{u_1(t)}{r^3}, & u_3(0) &= 0, \\
 Du_4(t) &= -\frac{u_2(t)}{r^3}, & u_4(0) &= \sqrt{\frac{1+e}{1-e}}, \quad r = \sqrt{u_1^2 + u_2^2}.
 \end{aligned}$$

Here e is the so-called eccentricity, which satisfies the condition $0 < e < 1$. Its exact solution is:

$$\begin{aligned}
 u_1(t) &= \cos E(t) - e, & u_2(t) &= \sqrt{1 - e^2} \sin E(t), \\
 u_3(t) &= \frac{\sin E(t)}{e \cos E(t) - 1}, & u_4(t) &= \frac{\sqrt{1 - e^2} \cos E(t)}{1 - \cos E(t)},
 \end{aligned}$$

where $E(t)$ is implicitly defined through as $E(t) - e \sin E(t)$. We solve the problem using the new "look-ahead" linear multistep methods in the interval $[0, 10]$ and the results obtained are shown in Table 8.

Table 8: Absolute errors in the numerical integration of example 7

t	u_i	Method (20) $ u(t) - u_n(t) $	Method (22) $ u(t) - u_n(t) $
5	u_1	$5.876743536248341 \times 10^{-12}$	$5.558664639693234 \times 10^{-12}$
	u_2	$1.662642557811143 \times 10^{-12}$	$1.563839811626042 \times 10^{-12}$
	u_3	$4.483768906290639 \times 10^{-18}$	$2.088306791896274 \times 10^{-18}$
	u_4	$3.486100297322992 \times 10^{-14}$	$1.554312234475219 \times 10^{-15}$
50	u_1	$7.199063567497888 \times 10^{-11}$	$6.809441899235935 \times 10^{-11}$
	u_2	$2.036728881184865 \times 10^{-11}$	$1.915699900551264 \times 10^{-11}$
	u_3	$7.053243351786559 \times 10^{-16}$	$3.284065321117538 \times 10^{-16}$
	u_4	$4.262146191535976 \times 10^{-13}$	$1.931788062847772 \times 10^{-14}$
250	u_1	$3.658250369298344 \times 10^{-10}$	$3.460272068878112 \times 10^{-10}$
	u_2	$1.034972160938682 \times 10^{-10}$	$9.734795795531860 \times 10^{-11}$
	u_3	$1.827310275748922 \times 10^{-14}$	$8.508022438695820 \times 10^{-15}$
	u_4	$2.164379786506743 \times 10^{-12}$	$9.869882688917642 \times 10^{-14}$
500	u_1	$7.331056872672548 \times 10^{-10}$	$6.934367524635832 \times 10^{-10}$
	u_2	$2.074054142949151 \times 10^{-10}$	$1.950846827153017 \times 10^{-10}$
	u_3	$7.341386342157169 \times 10^{-14}$	$3.418203491523802 \times 10^{-14}$
	u_4	$4.332756375902136 \times 10^{-12}$	$1.956212969389526 \times 10^{-13}$

Conclusion

The new work considered in the present paper is the construction of the recent introduced "look-ahead" linear multistep methods which extends some conventional multistep interpolation and collocation at the step point beyond the present step. In this way acceptable stability with characteristic roots of quadratic polynomial were obtained. These led to A-stable regions of absolute stability as

for the implicit Runge-Kutta methods [22]. As the first confirmation of their accuracy and stability properties, we have applied them on some set of initial value problems of ordinary differential equations available in literature. In our future work, we will discuss stability of look-ahead linear multistep collocation techniques that can achieve high-order including symmetry characterization. Additionally, we aim to apply related LALM

collocation techniques to other types of modeled differential equation problems, like cardiac electrophysiology and populations with tumor immune interactions.

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