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On the Properties and Applications of Sine Type Ii Topp-Leone Kumaraswamy Distribution

E. S. Chaku [1]; B.Maijama[1]; A. A. Makada[2], M. A. Abubakar [1]

[1]Department of Statistics and Data Analytics, Nasarawa State University Keffi, Nasarawa, Nigeria

[2]Department of Mathematics and Statistics, Niger State Polytechnic, Zungeru

Abstract

Statistical probability distributions constitute a fundamental component of data analysis by providing rigorous mathematical frameworks for modelling uncertainty and characterizing observed phenomena. However, many real-world datasets exhibit complex distributional features, including pronounced skewness, heavy-tailed behaviour, and non-monotonic hazard rate functions, which are often inadequately represented by conventional probability distributions. Addressing these challenges requires the development of more flexible statistical models capable of capturing diverse data structures while providing improved goodness-of-fit. In this study, a novel probability model, termed the Sine Type II Topp-Leone-Kumaraswamy (STII-TLKw) distribution, is proposed to accommodate complex data patterns encountered across a broad spectrum of scientific and engineering applications. The proposed distribution is formulated by integrating the flexibility of the Sine Type II generator with the Topp-Leone-Kumaraswamy family, resulting in a highly adaptable model with enhanced distributional properties. Its fundamental statistical characteristics, including the probability density function, cumulative distribution function, and other important mathematical properties, are derived and thoroughly investigated. Parameter estimation is performed using the Maximum Likelihood Estimation (MLE) method, while the finite-sample performance of the estimators is evaluated through an extensive Monte Carlo simulation study to assess their accuracy, consistency, and reliability. The practical applicability of the proposed distribution is further demonstrated using two real-life datasets. Model performance is assessed through standard information criteria and compared with several existing competing distributions. The STII-TLKw distribution consistently provides the best fit, yielding the lowest Akaike Information Criterion (AIC) values of 114.726 and 104.2888, together with the lowest Bayesian Information Criterion (BIC) values of 120.4631 and 109.9024, respectively. These findings demonstrate that the proposed distribution offers superior modelling flexibility and predictive capability for complex empirical data.

Keywords

Sine Type II Topp-Leone-G,
Kumaraswamy Distribution,
Survival function,
Hazard function,
Maximum likelihood,
Order Statistics.

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Faculty of Science, ATBU Bauchi

Corresponding Author: E. S. Chaku, Department of Statistics and Data Analytics, Nasarawa State University Keffi, Nasarawa, Nigeria

Email: makadaabdullah@gamil.com

Introduction

The Kumaraswamy distribution named after Poondi Kumaraswamy, and introduced in 1980, is a series of continuous probability distributions on the unit interval (0,1). Like the beta distribution, it has received a lot of attention in simulation literature due to its versatility and mathematical ease. Remarkably, it is computationally convenient as its probability density function, cumulative distribution function and quantile function are all available in closed-form expressions making the distribution appealing to practical applications.

The Kumaraswamy distribution has found much use in hydrology especially in the description of natural phenomena with limited values of the processes. Due to its analytical characteristics and simplicity of use, it offers an effective and simple modeling system, thus creating its applicability in hydrological research and other areas.

Over the last few years, much effort has been dedicated to an extension of the Kumaraswamy distribution in order to make it more flexible, and to increase its range of application both in the theoretical and practical realms. Examples of such extensions are the Kumaraswamy distribution which includes the generalized Kumaraswamy distribution by Carrasco et al., (2010), the exponentiated Kumaraswamy distribution by Lemonte et al. (2013), the Kumaraswamy-Kumaraswamy by El-Sherpieny et al. (2014), the Lomax-Kumaraswamy by Asiribo et al. (2018), the Generalized Exponential Kumaraswamy Distribution by NK et al., (2018), Gamma-Kumaraswamy Distribution by Ghosh and Hamedani (2018), Cubic Rank Transmuted Kumaraswamy Distribution by Saracoglu and Taniş (2018), Generalized Transmuted Kumaraswamy Distribution by Ishaq et al., (2019), Generalized Modification of the Kumaraswamy Distribution by Alshkaki (2020), Log-Kumaraswamy Distributions by Ishaq et al., (2023), and the Generalized Odd Maxwell-Kumaraswamy Distribution by Ishaq et al., (2024) On the whole, all these advancements further command the Kumaraswamy distribution as a powerful and multifaceted tool to model strenuous data in an extensive collection of scientific disciplines.

Isa et al. (2023) presented an innovative family of distributions called the Sine Type II Topp-Leone-G (STIITL-G) family of distribution which provides a better range of flexibility and better capability to describe the characteristics of various data sets. This distribution family is defined by a shape parameter, which is denoted by a and may be used to any arbitrary cumulative distribution function (cdf) $H\left(x,\xi\right)$. Cumulative distribution function (cdf) and probability density function (pdf) of STIITL-G are as follows.

$$F(x;\alpha,\xi)=\sin\left\{\frac{\pi}{2}\left[1-\left(1-H^2(x;\xi)\right)^{\alpha}\right]\right\},\quad \mathrm{x}>0,\alpha>0\,, \tag{1}$$

and

$$f(x;\alpha,\xi)=\frac{\pi}{2}2\alpha h(x;\xi)H(x;\xi)[1-H^2(x;\xi)]^{\alpha-1}\cos\left\{\frac{\pi}{2}\left[1-\left(1-H^2(x;\xi)\right)^{\alpha}\right]\right\} \tag{2}$$

The Cdf and Pdf of the Kumaraswamy Distribution Are Given As

$$H(x;\lambda,\beta)=1-\left[1-x^{\lambda}\right]^{\beta},\ 0<x<1,\ \lambda,\beta>0 \tag{3}$$

$$h(x;\lambda,\beta)=\lambda\beta x^{\lambda-1}\left[1-x^{\lambda}\right]^{\beta-1},\ 0<x<1,\ \lambda,\beta>0 \tag{4}$$

In this paper, we seek to increase the versatility of statistical modeling by deriving a new model based on Sine type II topp-Leone-Kumaraswamy distribution, which is called Sine Type II Topp-Leone-Kumaraswamy (STIITL-Kw) distribution.

The paper will be organized in the following way: Section 2 defines and introduces STIITL-Kw distribution. In section 3, the model presents some of the most important representations. Section 4 explores several statistical features such as probability-weighted moments, moments, the moment-generating function, the quantile function, the reliability function, the hazard function and order statistics. Section 5 dwells on parameter estimation applying the maximum likelihood estimation (MLE) technique. Section 6 includes the simulation study to determine the efficiency and consistency of MLE. In Section 7, the new model is used to work with real-life data to demonstrate its real-world usefulness. Lastly, there is conclusiveness in the form of a final remark made in Section 8.

1. Sine Type II Topp-Leone-Kumaraswamy (STIITL-Kw) Distribution

In the present section, we present a new model that is known as STIITL-Kw distribution. An X which is a random variable is said to obey the STIITL-Kw distribution when the cumulative distribution function (cdf) of the random variable X is obtained by replacing Equation (3) with Equation (1) and this leads to X assuming the following expression:

$$F(x;\alpha,\lambda,\beta)=\sin\left\{\frac{\pi}{2}\left[1-\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]^{\alpha}\right]\right\},\quad \mathrm{x}>0,\alpha,\lambda,\beta>0 \tag{5}$$

The STIITL-Kw distribution can be differentiated to give the following as the pdf:

$$f(x;\alpha,\lambda,\beta)=\frac{\pi}{2}2\alpha\lambda\beta x^{\lambda-1}\left[1-x^{\lambda}\right]^{\beta-1}\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]^{\alpha-1} \cos\left\{\frac{\pi}{2}\left[\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]^{\alpha}\right]\right\} \tag{6}$$

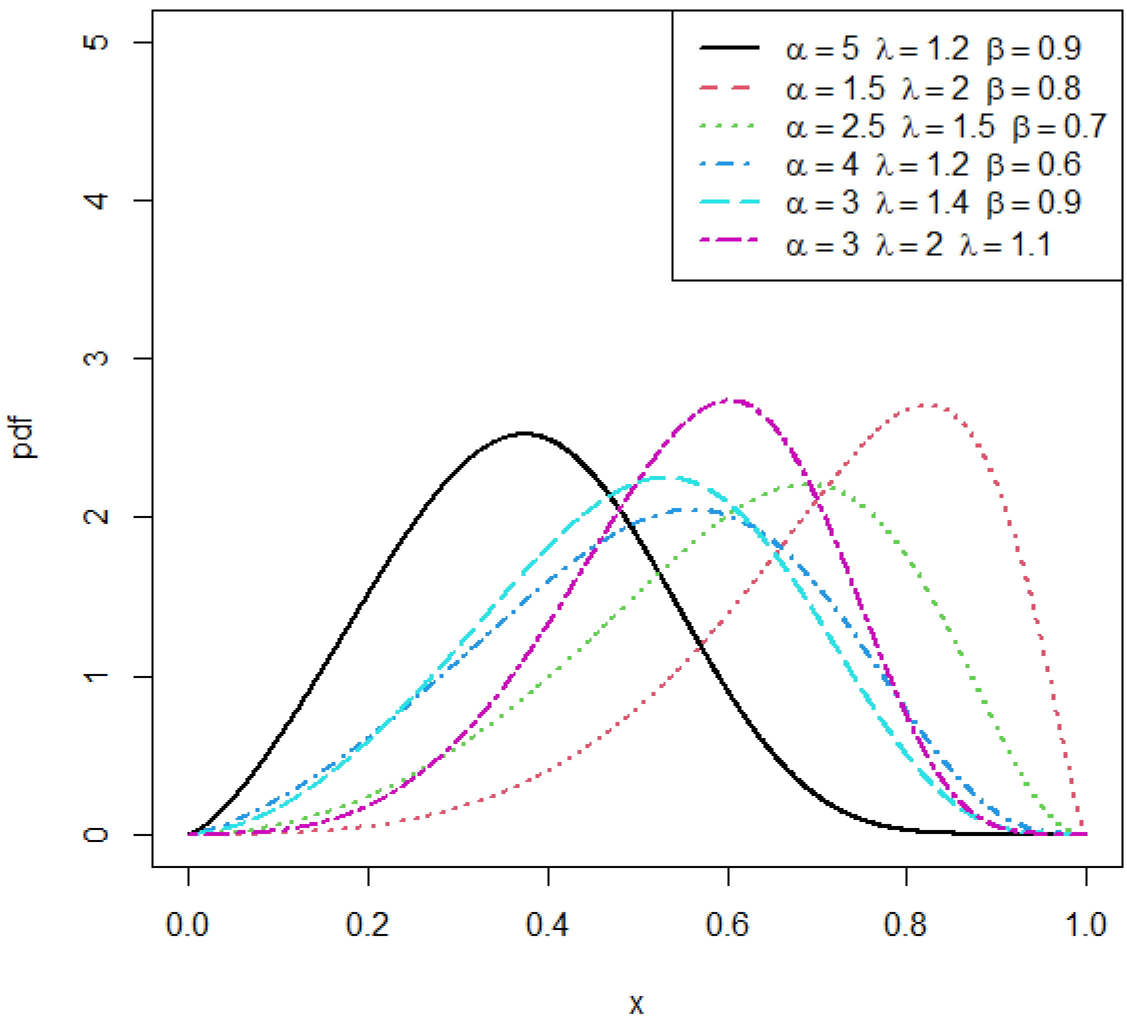

Figure 3.1: Plot of pdf of STIITL-Kw distribution for different values of parameters.

2. Suitable Expansion of Density for STIITL-Kw Distribution

In this part, we obtain the suitable expressions of the pdf and cdf of the STIITL-Kw distribution. The expressions are developed taking the Maclaurin series expansions for the sine and cosine functions and binomial expansion methods as listed below:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{7}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n!)} \tag{8}$$

$$1-y^{b}=\sum_{k=0}^{\infty}(-1)^{k}\binom{b}{k}y^{k} \tag{9}$$

The suitable expression for density in the STIITL-Kw is calculated using the Maclaurin series expansion of the cosine function, as well as the binomial expansion, as given in equation (8) and equation (9), and used them in equation (6).

$$\cos\left\{\frac{\pi}{2}\left[1-\left(1-[1-x^{\alpha}]^{2}\right)^{r}\right]\right\}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\pi^{2i}}{2^{2i}(2i)!}\left[1-\left(1-[1-x^{\alpha}]^{2}\right)^{r}\right]^{2i}$$

$$\left[1-(1-x^{\alpha})^{2}\right]^{r(2i)}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\alpha(i+1)-1}{j}[1-x^{\alpha}]^{2rj}$$

Therefore, the suitable expansion for the pdf can be expressed as

$$f(x;\alpha,\lambda,\beta)=2\alpha\lambda\beta x^{\alpha-1}\sum_{i,j,m=0}^{\infty}\frac{(-1)^{i+j+m}\pi^{2i}\binom{\alpha(i+1)-1}{j}\binom{2j+1}{m}}{(2i)!2^{2i}}[1-x^{\alpha}]^{m+\alpha(i+1)-1} \tag{10}$$

Also, the suitable expansion for the CDF of the STIITLKw can be expressed as follows:

$$\sin\left\{\frac{\pi}{2}\left[1-\left(1-[1-(1-x^{\alpha})^{2}]^{r}\right)\right]\right\}^{d}=\sum_{d=0}^{\infty}\frac{(-1)^{d}\pi^{2d+1}}{(2d+1)!2^{2d+1}}\left[1-\left(1-[1-(1-x^{\alpha})^{2}]^{r}\right)\right]^{d}$$

$$\left[1-\left(1-[1-x^{\alpha}]^{2}\right)^{r}\right]^{d}=\sum_{b=0}^{\infty}(-1)^{b}\binom{d}{b}\left[1-[1-x^{\alpha}]^{2}\right]^{2rb}$$

$$F(x;\alpha,\lambda,\beta)=\sum_{d,b,j,r=0}^{\infty}\frac{(-1)^{d+b+j+r}\binom{d}{b}\binom{ab}{j}\binom{2j}{r}\pi^{2d+1}}{(2d+1)!2^{2d+1}}[1-x^{\alpha}]^{r\alpha} \tag{11}$$

3. Statistical Properties of STIITL-Kw Distribution

In this section we will find out the statistical properties of the STIITL-Kw distribution.

3.1 Moments of STIITL-Kw Distribution

Moments are important in statistical analysis especially in practice. Thus, we obtain the r^{th} moment of the STIITL-Kw distribution.

$$E(X^{r})=\int_{0}^{1}x^{r}f(x)dx \tag{12}$$

$$E(X^{r})=2\alpha\lambda\beta x^{\lambda-1}\sum_{i,j,m=0}^{\infty}\frac{(-1)^{i+j+m}\pi^{2i+1}\binom{\alpha(i+1)-1}{j}\binom{2j+1}{m}}{(2i)!2^{2i+1}}\int_{0}^{1}x^{r}\left[1-x^{\lambda}\right]^{\beta(m+1)-1}dx \tag{13}$$

Consider the integral part in equation (13)

Consider the integral part in equation (13)

$$\int_0^1 x^r \left[1-x^\lambda\right]^{\beta(m+1)-1} dx$$

Let $y=\left[1-x^\lambda\right] \Rightarrow x= 1-y^{\frac{1}{\lambda}}$; when $x \Rightarrow 0, y \Rightarrow 1$; when $x \Rightarrow 1, y \Rightarrow 0$; $dx=\frac{-dy}{\lambda x^{\lambda-1}}$

Then

$$E(X^r)=2\alpha\lambda\beta x^{\lambda-1}\sum_{i,j,m=0}^{\infty}\frac{(-1)^{i+j+m}\pi^{2i+1}\binom{\alpha(i+1)-1}{j}\binom{2j+1}{m}}{(2i)!2^{2i+1}}\int_0^1 \left[1-y\right]^{\frac{r}{\lambda}}\left[y\right]^{\beta(m+1)-1}\frac{dy}{\lambda x^{\lambda-1}}$$

where $\int_0^1 \left[1-y\right]^{\frac{r}{\lambda}} y^{\beta(m+1)-1}dy = B\left(1+\frac{r}{\lambda},\beta(m+1)\right)$

Therefore

$$E(X^r)=2\alpha\beta\sum_{i,j,m=0}^{\infty}\frac{(-1)^{i+j+m}\pi^{2i+1}\binom{\alpha(i+1)-1}{j}\binom{2j+1}{m}}{(2i)!2^{2i+1}}B\left(1+\frac{r}{\lambda},\beta(m+1)\right) \tag{14}$$

The equation given above (14) is the r^{th} moment of the STIITL-Kw distribution.

3.2 Moment generating function (mgf) of STIITL-Kw Distribution

The Moment Generating Function is given as:

$$M_x(t)=\int_0^1 e^{tx} f(x)dx \tag{15}$$

The mgf of the STIITL-Kw distribution can be obtained by replacing equation (10) by equation (15), and the following expression is obtained.

$$M_x(t)=2\alpha\lambda\beta x^{\lambda-1}\sum_{i,j,m=0}^{\infty}\frac{(-1)^{i+j+m}\pi^{2i+1}\binom{\alpha(i+1)-1}{j}\binom{2j+1}{m}}{(2i)!2^{2i+1}}\int_0^1 e^{tx}\left[1-x^\lambda\right]^{\beta(m+1)-1}dx \tag{16}$$

By expanding $e^{tx}=\sum_{b=0}^{\infty}\frac{t^b x^b}{b!}$ and following the process for deriving moments as outlined above, we obtain the mgf for the STIITL-Kw distribution, which is presented in equation (17) below.

$$M_x(t)=2\alpha\beta\frac{t^b}{b!}\sum_{b=0}^{\infty}\sum_{i,j,m=0}^{\infty}\frac{(-1)^{i+j+m}\pi^{2i+1}\binom{\alpha(i+1)-1}{j}\binom{2j+1}{m}}{(2i)!2^{2i+1}}B\left(1+\frac{b}{\lambda},\beta(m+1)\right) \tag{17}$$

3.3 Reliability Function of STIITL-Kw Distribution

The reliability function, also referred to as the survival function, represents the probability that an item does not fail before a given time. It is defined as follows:

$$R(x;\alpha,\lambda,\beta)=1-F(x;\alpha,\lambda,\beta)$$

(18)

$$R(x;\alpha,\lambda,\beta)=1-\sin\left\{\frac{\pi}{2}\left[1-\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]^{\alpha}\right]\right\} \tag{19}$$

3.4 Hazard Function of STIITL-Kw Distribution

The hazard function describes how the risk of an event evolves over time. It represents the failure rate or the instantaneous rate of occurrence of the event at a specific time, given that the event has not occurred up to that time. It is defined as:

$$T(x;\alpha,\lambda,\beta)=\frac{f(x;\alpha,\lambda,\beta)}{R(x;\alpha,\lambda,\beta)} \tag{20}$$

$$T(x;\alpha,\lambda,\beta)=\frac{\frac{\pi}{2}\,2\alpha\lambda\beta\,x^{\alpha-1}\left[1-x^{\alpha}\right]^{\beta-1}\left[1-(1-x^{\alpha})^{\beta}\right]^{\lambda-1}\left\{1-\left[1-(1-x^{\alpha})^{\beta}\right]^{\lambda}\right\}^{2\gamma-1}\cos\Bigg(\frac{\pi}{2}\Bigg\{1-\Bigg[1-\Bigg\{1-(1-x^{\alpha})}{1-\sin\left(\frac{\pi}{2}\left\{1-\left[1-\left\{1-(1-x^{\alpha})^{\beta}\right\}^{\lambda}\right]^{2\gamma}\right\}\right)} \tag{21}$$

3.5 Quantile Function of STIITL-Kw Distribution

The quantile function is an essential tool for generating random variables from any continuous probability distribution, making it highly significant in probability theory. For a given x, the quantile function is defined as F(x) = u, where u follows a U (0, 1) distribution. The quantile function, also known as the inverse cumulative distribution function (cdf), for the STIITL-Kw distribution is derived using the CDF provided in equation (5).

$$x=Q(u)=\left[1-\left[1-\left[1-\left[1-\frac{\sin^{-1}(U)}{\pi/2}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2}}\right]^{\frac{1}{\beta}}\right]^{\frac{1}{\lambda}} \tag{22}$$

The median of the STIITL-Kw distribution can be obtained by substituting u = 0.5 into equation (22), as shown below.

$$\text{median} = Q(0.5) = \left[1-\left[1-\left[1-\left[1-\frac{\sin^{-1}(0.5)}{\pi/2}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2}}\right]^{\frac{1}{\beta}}\right]^{\frac{1}{\lambda}} \tag{23}$$

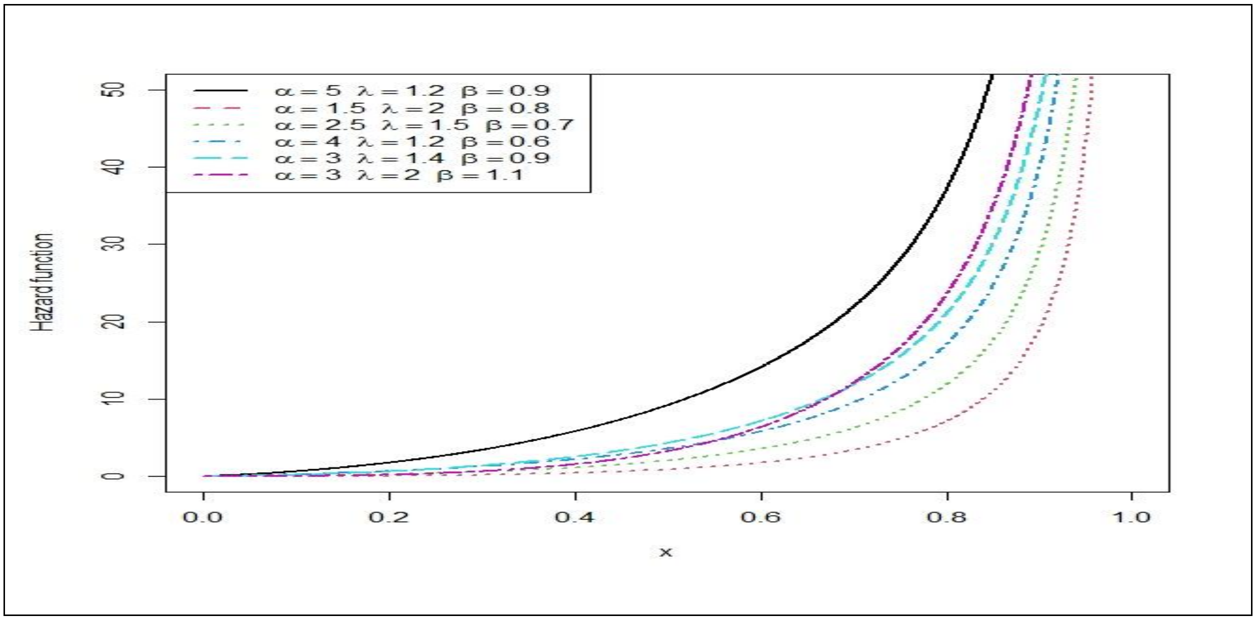

Figure 2: Plot of reliability function of the STIITL-Kw distribution for different valves of parameters

3.6 Distribution of Order Statistics of STIITL-Kw Distribution

Let $x_1, x_2, ..., x_3$ be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function $F(x)$. Let $X_{(1:n)} < X_{(2:n)} < ... < X_{(n:n)}$ the corresponding ordered random sample from the STIITL-Kw distributions. Let $F_{k:n(x)}$ and $f_{k:n(x)}$, $k = 1,2,3,...,n$ denote the CDF and PDF of the k^{th} order statistics $X_{k:n}$ respectively. The PDF of the k^{th} order statistics of $X_{k:n}$ is given as

$$f_{k:n}(x;\alpha,\lambda,\beta) = \frac{f(x)}{B(k,n-k+1)}\sum_{v=0}^{n-k}(-1)^v\binom{n-k}{v}\left[F(x)\right]^{v+k-1} \tag{24}$$

The PDF of the r[th] order statistic for the STIITL-Kw distribution is derived by substituting equations (10) and (11) into equation (24).

$$f_{k:n}(x;\alpha,\lambda,\beta)=\sum_{v=0}^{n-k}\sum_{i,j,m=0}^{\infty}\sum_{d,b,l,r=0}^{\infty}\frac{2\alpha\lambda\beta x^{\lambda-1}(-1)^{i+j+m+v+d+b+l+r}\pi^{2i+1+2d+1}\binom{n-k}{v}\binom{\alpha(i+1)-1}{j}\binom{2j+1}{m}\binom{d}{b}\binom{\alpha b}{l}\binom{2l}{r}}{B(k,n-k+1)(2d+1)!(2i)!2^{2i+1+2d+1}}\left[1-x^{\lambda}\right] \tag{25}$$

The PDF of the minimum order statistic for the STIITLKw distribution is derived by setting k = 1 in equation (25), yielding the following expression.

$$f_{1:n}(x;\alpha,\lambda,\beta)=\sum_{v=0}^{n-1}\sum_{i,j,m=0}^{\infty}\sum_{d,b,l,r=0}^{\infty}\frac{2n\alpha\lambda\beta x^{\lambda-1}(-1)^{i+j+m+v+d+b+l+r}\pi^{2i+1+2d+1}\binom{n-1}{v}\binom{\alpha(i+1)-1}{j}\binom{2j+1}{m}\binom{d}{b}\binom{\alpha b}{l}\binom{2l}{r}}{(2d+1)!(2i)!2^{2i+1+2d+1}}\left[1-x^{\lambda}\right]^{\beta(m+rv+1)-1} \tag{26}$$

Likewise, the PDF of the maximum order statistic for the STIITLKw distribution is obtained by setting k = n in equation (25), yielding the following expression.

$$f_{n:n}(x;\alpha,\lambda,\beta)=\sum_{i,j,m=0}^{\infty}\sum_{d,b,l,r=0}^{\infty}\frac{2n\alpha\lambda\beta x^{\lambda-1}(-1)^{i+j+m+v+d+b+l+r}\pi^{2i+1+2d+1}\binom{\alpha(i+1)-1}{j}\binom{2j+1}{m}\binom{d}{b}\binom{\alpha b}{l}\binom{2l}{r}}{(2d+1)!(2i)!2^{2i+1+2d+1}}\left[1-x^{\lambda}\right]^{\beta(m+1)-1+\beta r(v+n-1)} \tag{27}$$

4. Maximum Likelihood Estimation of STIITL-Kw Distribution

The maximum likelihood estimation method is used in estimating the parameters of the new model. Let $x_1, x_2, x_3,..., x_n$ be a random sample of size n from the STIITL-Kw distribution. Then, the likelihood function based on the observed sample for the parameter vector $(\alpha,\lambda,\beta)^T$ is given by:

$$\begin{aligned}\log(L)=n\log\left(\frac{\pi}{2}\right)+n\log(2)+n\log(\alpha)+n\log(\lambda)+n\log(\beta)+(\lambda-1)\sum_{i=1}^{n}\log(x_i)\\+(\beta-1)\sum_{i=1}^{n}\log\left[1-x_i^{\lambda}\right]+\sum_{i=1}^{n}\log\left[1-\left[1-x_i^{\lambda}\right]^{\beta}\right]\\+(\alpha-1)\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]+\sum_{i=1}^{n}\log\left[\cos\left[\frac{\pi}{2}\left[\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]^{\alpha}\right]\right]\right]\end{aligned} \tag{28}$$

Differentiating the log-likelihood function with respect to α,λ,β and setting the result to zero, we obtain:

$$\frac{\partial L}{\partial \alpha}=\frac{n}{\alpha}+\tan\left[\frac{\pi}{2}\left[\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]^{\alpha}\right]\right]\left[\frac{\pi}{2}\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]^{\alpha}\log\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]\right] \tag{29}$$

$$\begin{aligned}\frac{\partial L}{\partial \lambda}=&\frac{n}{\lambda}+\sum_{i=1}^{n}\log x_i-\beta-1\sum_{i=1}^{n}\frac{x_i^{\lambda}\log x_i}{\left[1-x_i^{\lambda}\right]}+\sum_{i=1}^{n}\frac{\beta\left[1-x_i^{\lambda}\right]^{\beta-1}x_i^{\lambda}\log x_i}{\left[1-\left[1-x_i^{\lambda}\right]^{\beta}\right]}\\&-\alpha-1\sum_{i=1}^{n}\frac{2\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]\beta\left[1-x_i^{\lambda}\right]^{\beta-1}x_i^{\lambda}\log x_i}{\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]}\\&+\sum_{i=1}^{n}\tan\left[\frac{\pi}{2}\left[\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]^{\alpha}\right]\right]\left[\frac{\pi}{2}\alpha\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]^{\alpha-1}2\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]\beta\left[1-x_i^{\lambda}\right]^{\beta-1}x_i^{\lambda}\log x_i\right]\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\partial L}{\partial \beta}=&\frac{n}{\beta}+\sum_{i=1}^{n}\log\left[1-x_i^{\lambda}\right]-\sum_{i=1}^{n}\frac{\left[1-x_i^{\lambda}\right]^{\beta}\log\left[1-x_i^{\lambda}\right]}{\left[1-\left[1-x_i^{\lambda}\right]^{\beta}\right]}+\alpha-1\sum_{i=1}^{n}\frac{2\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]\left[1-x^{\lambda}\right]\log\left[1-x^{\lambda}\right]}{\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]}\\&+\sum_{i=1}^{n}\tan\left[\frac{\pi}{2}\left[\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]^{\alpha}\right]\right]\left[\frac{\pi}{2}\alpha\left[1-\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]^{2}\right]^{\alpha-1}2\left[1-\left[1-x^{\lambda}\right]^{\beta}\right]\left[1-x^{\lambda}\right]^{\beta}\log\left[1-x^{\lambda}\right]\right]\end{aligned} \tag{31}$$

The equations (29), (30), and (31) presented above are nonlinear and cannot be solved analytically.

5. Simulation Study of STIITL-Kw Distribution

This part attempts to review the reliability of the parameter estimates of the proposed STIITL-Kw distribution by using a detailed simulation study. With Monte Carlo simulations, we determined the bias and root mean square error (RMSE) of the maximum likelihood estimates (MLEs) of the parameters. Simulated data were also created with different sample sizes (n = 20, 50, 100, 250, 500 and 1,000) with the use of the log-likelihood function, as in Equation (28) and the quantile function, as in Equation (22). The simulations were run 1000 times on each sample and the parameter values that were fixed were ($a=1.3$, $b=1.1$, $l=1.5$ and $a=1.7$, $b=1.5$, $l=2$) in the case of STIITL-Kw.

It was an effective way to evaluate the consistency of the estimation process and the strength of the parameter estimates at varying sample sizes.

Table 1: MLEs, biases and RMSE for some values of parameters

		(1.5,1.3,1.1)			(2,1.7,1.5)		
n	Parameters	Estimated Values	Bais	RMSE	Estimated Values	Bais	RMSE
20	λ	1.6984	0.1984	0.5386	2.2189	0.2189	0.6095
	α	1.4919	0.1919	0.6779	1.9638	0.2638	0.8761
	β	1.3039	0.2039	0.5674	1.7810	0.2810	0.7743
50	λ	1.5769	0.0769	0.3028	2.0866	0.0866	0.3463
	α	1.3974	0.0974	0.5208	1.8000	0.1000	0.6111
	β	1.2071	0.1071	0.4060	1.6539	0.1539	0.5214
100	λ	1.5468	0.0468	0.2030	2.0477	0.0477	0.2324
	α	1.3303	0.0303	0.4111	1.7528	0.0528	0.4945
	β	1.1819	0.0819	0.3045	1.5921	0.0921	0.3858
250	λ	1.5140	0.0140	0.1233	2.0151	0.0151	0.1427
	α	1.3422	0.0422	0.3143	1.7540	0.0540	0.3725
	β	1.1208	0.0208	0.2080	1.5219	0.0219	0.2577
500	λ	1.5049	0.0049	0.0842	2.0054	0.0054	0.0975
	α	1.3161	0.0161	0.2270	1.7186	0.0186	0.2790
	β	1.1134	0.0134	0.1522	1.5173	0.0173	0.1951
1000	λ	1.5012	0.0012	0.0607	2.0016	0.0016	0.0717
	α	1.3090	0.0090	0.1682	1.7110	0.0110	0.2160
	β	1.1073	0.0073	0.1125	1.5103	0.0103	0.1502

The table indicates that an increase in the sample size results in a reduction in the bias and RMSE values to a zero point. The trend shows that the estimates are more accurate and closer to the actual values; hence they are efficient and reliable.

6. Application of STIITL-Kw to Real-life Datasets

The section shows clearly the versatility and flexibility of the Sine Type II Topp-Leone Kumaraswamy (STIITL-Kw) distribution because it has been applied to two real-life datasets. Kumaraswamy-

Kumaraswamy (Kw-Kw) Distribution by El-Sherpieny et al., (2014), the Weibull-Kumaraswamy (W-kw) Distribution by Aminu et al., (2018), the Type II Half Logistic Kumaraswamy (TIIHL-Kw) Distribution by ZeinEldin et al., (2020), Exponentiated Kumaraswamy (E-Kw) distribution by Lemonte et al., (2013), and the Kumaraswamy (Kw) Distribution by Kumaraswamy, (1980).
Kumaraswamy-Kumaraswamy (Kw-Kw) distribution (El-Sherpieny et al., 2014).

$$f(x;\lambda,\beta,\alpha,\theta)=\lambda\beta\alpha\theta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\theta-1}\left(1-\left(1-x^{\alpha}\right)^{\theta}\right)^{\lambda-1}\left(1-\left(1-\left(1-x^{\alpha}\right)^{\theta}\right)^{\lambda}\right)^{\beta-1}$$

Weibull-Kumaraswamy (Wkw) distribution (Aminu et al., 2018)

$$f(x;\lambda,\beta,\alpha,\theta)=\lambda\theta\alpha\beta\frac{x^{\lambda-1}\left(1-x^{\lambda}\right)^{\theta-1}}{\left[1-\left(1-x^{\lambda}\right)^{\theta}\right]}\left[-\log\left(1-\left(1-x^{\lambda}\right)^{\theta}\right)\right]^{\beta-1}\exp\left(-\alpha\left(-\log\left(1-\left(1-x^{\lambda}\right)^{\theta}\right)\right)^{\beta}\right)$$

Type II Half Logistic Kumaraswamy (TIIHLKw) distribution (ZeinEldin et al., 2020)

$$f(x;\lambda,\beta,\alpha)=\frac{2\lambda\alpha\beta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta-1}\left(1-\left(1-x^{\alpha}\right)^{\beta}\right)^{\lambda-1}}{\left(1+\left(1-\left(1-x^{\alpha}\right)^{\beta}\right)^{\lambda}\right)^{2}}$$

Exponentiated Kumaraswamy (EKw) distribution (Lemonte et al., 2013)

$$f(x;\beta,\alpha,\theta)=\alpha\beta\theta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta-1}\left(1-\left(1-x^{\alpha}\right)^{\beta}\right)^{\theta-1}$$

Kumaraswamy (Kw) distribution (Kumaraswamy, 1980)

$$f(x;\alpha,\theta)=\alpha\theta x^{\alpha-1}\left[1-x^{\alpha}\right]^{\theta-1}$$

These two sets of data applied to show how the new distribution can be applied give concrete data on the flexibility and appropriateness of the new distribution. It is demonstrated to be the most suitable in modelling the datasets and is better than the other comparator distributions. The R programming language was used to do all calculations.

Dataset 1

The first dataset, obtained from Dasgupta (2011), contains 50 measurements of burr size (in millimeters) for a hole with a 12 mm diameter and a sheet thickness of 3.15 mm.
0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16.

Table 2: The MLEs, Log-likelihoods and Goodness of Fits Statistics of the models based on Dataset 1

Model	l	α	b	θ	LL	AIC	BIC
STIITL-Kw	1.2957	4.1943	3.8085	-	-54.3635	114.727	120.4631
Kw-Kw	0.4756	4.0125	5.0524	23.6156	-56.6240	121.2480	128.8961
W-kw	1.6347	3.7808	4.1275	17.7379	-55.2668	118.5336	126.1817
TIIHL-Kw	0.2948	6.1734	0.7553	-	-57.4506	120.9012	126.6373
E-Kw	-	1.9621	25.4879	0.9594	-55.8462	117.6924	123.4285
Kw	-	1.8553	-	22.1313	-58.5953	121.1906	125.0146

The Maximum Likelihood Estimation (MLE) of the parameters of the STIITL-Kw distribution as well as five competing distributions is shown in Table 2. The STIITL-Kw distribution had the lowest values of AIC and BIC, 114.727 and 120.4631, respectively. It means that STIITL-Kw distribution provides the most suitable approach to the burr measurements (in millimeters) with hole diameter of 12 mm and the sheet thickness of 3.15 mm, when compared to the other distributions considered in the analysis.

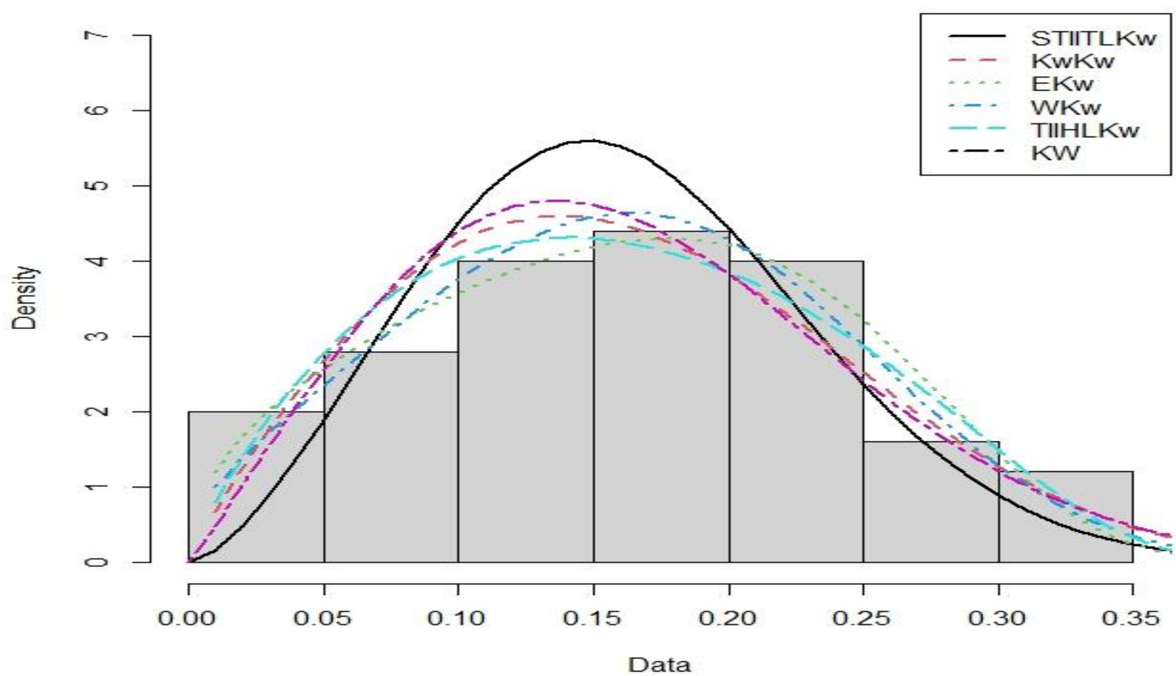

Figure 3: Fitted pdfs for the STIITL-Kw, Kw-Kw, E-Kw, W-Kw, TIIHL-Kw and K-w models to the burr (in millimeters), with a hole diameter of 12 mm and a sheet thickness of 3.15 mm data.

Figure 3 is the visual assessment of the model fit, which shows that STIITL-Kw distribution has better performance compared to the rest of the distributions. The graphical display extends its usefulness in providing an accurate modeling to the data, and it is further corroborated by the statistical results of the best fit of AIC and BIC values that it is the best of the distributions considered.

Dataset 2

The second dataset, sourced from Cordeiro and dos Santos Brito (2012), includes 48 measurements collected from petroleum rock samples within a petroleum reservoir.

0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470.

Table 3: The MLEs, Log-likelihoods and Goodness of Fits Statistics of the models based on Dataset 2

Model	l	α	j	θ	LL	AIC	BIC
STIITL-Kw	1.4524	3.1733	3.7811	-	-49.1444	104.2888	109.9024
Kw-Kw	3.3765	1.1547	4.1536	4.6567	-54.5507	117.1014	124.5862
W-kw	0.6660	1.3824	4.4374	1.3154	-58.4117	124.8234	132.3082
TIIHL-Kw	3.6095	1.0428	5.6765	-	-50.8321	107.6642	113.2778
E-Kw	-	0.9957	7.1632	3.8152	-51.2206	108.4412	114.0548
K-w	-	2.3650	-	27.1709	-52.6831	109.3662	113.1086

The results of the Maximum Likelihood estimations of STIITL-Kw distribution and five other models are presented in Table 3. Out of all the models considered, STIITL-Kw distribution had the lowest values of (AIC = 104.2888) and (BIC = 109.9024). These findings show that the STIITL-Kw model is most suitable to describe the data of the petroleum rock sample recovered out of the reservoir dataset. The AIC and BIC values are much lower, indicating that this distribution would best represent and give an efficient representation to the observed data when compared to the other statistical models of consideration.

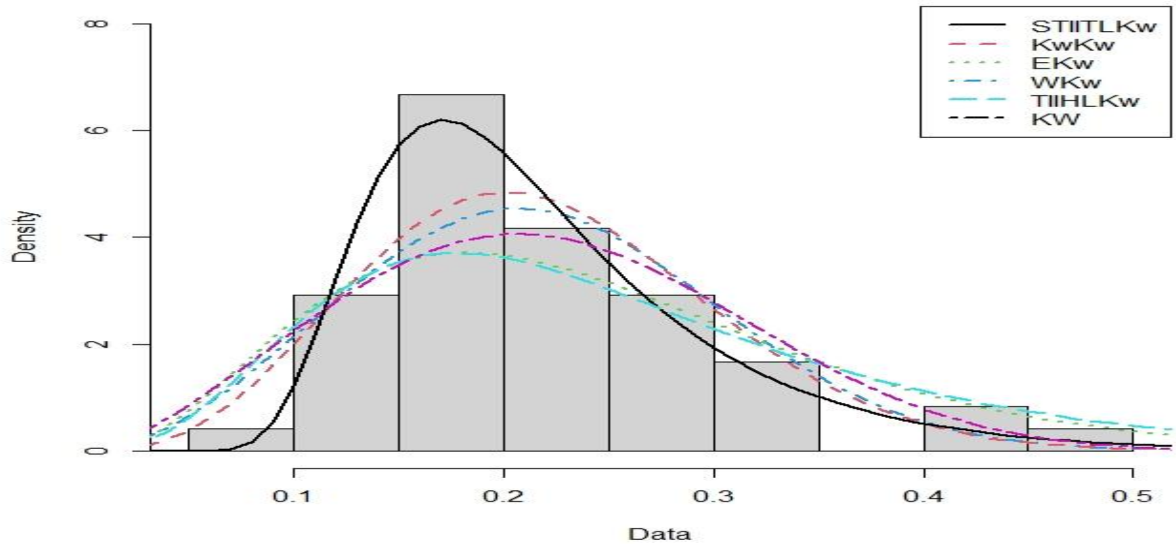

Figure 4: Fitted pdfs for the STIITL-Kw, Kw-Kw, E-Kw, W-Kw, TIIHL-Kw and K-w models to the petroleum rock sample measurements from a petroleum reservoir data

Figure 4 demonstrates an excellent performance of the STIITL-Kw distribution as compared to the other models, which is supported by the visual evaluation. It is known that the STIITL-Kw distribution provides the optimal fit to the petroleum rock sample measurements of the reservoir dataset of the distributions studied.

7. Conclusion

This research underlines the need to bring about the flexible probability distributions to the reality of science in order to capture the true complexity of the real-life data. In that regard, we developed the Sine Type II Topp-Leone-Kumaraswamy distribution, yet one more extension of the distribution family suggested by Isa et al. (2023) that is capable of fitting various data patterns. The basic statistical characteristics of the suggested distribution were obtained and the estimation of the parameters was conducted according to the maximum likelihood estimation tool. Besides, simulation studies have been done to assess the performance and reliability of the proposed model. The practical applicability of the new distribution was evidenced by using it on two real world data sets, where its performance was benchmarked against other established competitive models. It was revealed that the proposed distribution always fits better, which validated its flexibility and strength. The results of the present research support the idea that Sine Type II Topp-Leone-Kumaraswamy distribution is a powerful and useful tool in data modeling and can be effectively used in the broad scope of practical and scientific applications.

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