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Numerical Investigation of the Classical Blasius Boundary Layer Equation Using Similarity Transformation

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Abstract

This study investigates the classical Blasius boundary layer problem governing steady, incompressible, two-dimensional laminar flow over a stationary semi-infinite flat plate under zero pressure gradient conditions. The governing boundary layer equations are formulated from the Navier–Stokes equations by invoking the standard boundary layer approximations, which reduce the full fluid motion equations to a simplified system describing momentum transport within the viscous region adjacent to the plate. To facilitate numerical solution, the coupled partial differential equations are transformed into a nonlinear third-order ordinary differential equation through the application of the Blasius similarity transformation, thereby reducing the problem to a self-similar boundary value formulation. The resulting Blasius equation is solved numerically using the shooting technique in conjunction with the fourth-order Runge–Kutta integration scheme, enabling accurate determination of the unknown initial conditions that satisfy the far-field boundary constraints. Numerical simulations are performed to examine the dimensionless velocity profile, boundary layer growth, and wall shear characteristics along the flat plate. The computed results demonstrate smooth convergence of the velocity field toward the free-stream condition and exhibit excellent agreement with the well-established classical value of the skin-friction parameter, thereby validating the accuracy, stability, and reliability of the adopted numerical procedure. The findings reaffirm the effectiveness of similarity-based numerical techniques for solving nonlinear boundary layer problems and provide a robust computational framework for future investigations involving more complex transport phenomena, including coupled heat and mass transfer, magnetohydrodynamic flows, porous media, thermal radiation, viscous dissipation, chemical reactions, and non-Newtonian fluid models relevant to advanced engineering and industrial applications.

Keywords
Blasius Equation,
Boundary Layer Flow,
Similarity Transformation,
Nonlinear Ordinary Differential Equation,
Flat Plate,
Numerical Solution.

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1. INTRODUCTION

The study of viscous boundary layer flow is one of the fundamental areas of fluid mechanics. When a fluid moves over a solid surface, viscous effects become significant within a thin region adjacent to the surface known as the boundary layer. The concept of boundary layer was introduced by Prandtl and has become essential in engineering applications such as aerodynamics, heat exchangers, lubrication systems, and energy transport processes.

The exact solution of the laminar boundary layer flow over a flat plate was obtained by Blasius. The Blasius formulation transformed the governing partial differential equations into a nonlinear ordinary differential equation through the introduction of a suitable similarity variable.

Boundary layer theory remains one of the most significant developments in classical fluid mechanics, introduced by Prandtl in 1904 to explain viscous effects in high Reynolds number flows. In such flows, viscous forces are confined to a thin region near solid boundaries, while the outer region can often be approximated as inviscid. This conceptual framework has since become fundamental in the analysis of aerodynamic drag, heat transfer, and mass transport in engineering systems. A cornerstone of boundary layer theory is the classical laminar flow solution over a semi-infinite flat plate, first derived by Blasius in 1908. The resulting similarity solution reduces the governing partial differential equations into a third-order nonlinear ordinary differential equation now widely known as the Blasius equation. This formulation is remarkable because it demonstrates that the velocity field in the boundary layer can be expressed as a universal function independent of the streamwise coordinate when properly scaled.

The Blasius solution has been extensively studied due to its role as a benchmark for validating numerical schemes in viscous flow analysis. Schlichting and Gersten (2017) highlighted its importance in predicting wall shear stress and boundary layer thickness, while White (2016) emphasized its relevance in engineering applications involving aerodynamic surfaces. Kundu and Cohen (2015) further extended its interpretation in the context of modern fluid mechanics, particularly in relation to scaling laws and similarity transformations.

From a mathematical perspective, the transformation of the Navier–Stokes equations into the Blasius ordinary differential equation is achieved through the introduction of a similarity variable, which collapses the two-dimensional velocity field into a single independent variable. This reduction not only simplifies the governing equations but also provides deep insight into the self-similar nature of laminar boundary layers. Cebeci and Bradshaw (2005) demonstrated that such similarity solutions serve as foundational test cases for more complex numerical methods used in computational fluid dynamics.

Despite the availability of closed-form asymptotic approximations, the Blasius equation does not admit an exact analytical solution in elementary functions, necessitating numerical approaches. Consequently, methods such as the shooting technique combined with Runge–Kutta integration have become standard tools for obtaining accurate solutions, particularly for determining the wall shear parameter, which directly influences the skin friction coefficient.

In recent decades, the classical problem has also served as a baseline for extended studies involving magnetohydrodynamics, porous media, non-Newtonian fluids, and heat and mass transfer effects. These generalizations highlight the continuing relevance of the Blasius framework in both theoretical and applied fluid mechanic's research.

The resulting equation:

$f''' + 1/2\, ff'' = 0$

has become a benchmark problem for testing analytical and numerical methods for nonlinear differential equations.

2.0 MATHEMATICAL FORMULATION

Consider a steady, incompressible, two-dimensional flow over a flat plate.

The governing equations are:

Continuity equation:

$\partial u/\partial x + \partial v/\partial y = 0$

Momentum equation:

$u(\partial u/\partial x) + v(\partial u/\partial y)$

$\nu(\partial^2 u/\partial y^2)$

where u and v represent the velocity components in the x and y directions respectively, and ν is the kinematic viscosity.

The boundary conditions are:

At the wall:

$u = 0, \quad v = 0, \quad y = 0$

Far from the plate:

$u \rightarrow U\infty, \quad y \rightarrow \infty$

SIMILARITY TRANSFORMATION

A stream function is introduced:

$u = \partial\psi/\partial y$

$v = -\partial\psi/\partial x$

The similarity variables are defined as:

$\eta = y\sqrt{(U\infty/\nu x)}$

$\psi = \sqrt{(\nu U\infty x)}\ f(\eta)$

Therefore:

$u = U\infty f'(\eta)$

Substitution into the momentum equation gives:

$f''' + 1/2ff'' = 0$

This is the classical Blasius nonlinear ordinary differential equation.

The transformed boundary conditions become:

$f(0)=0$

$f'(0)=0$

$f'(\infty)=1$

3. MATERIALS AND METHOD

The Blasius equation is a third-order nonlinear equation and does not have a closed-form solution. Therefore, the equation is converted into a system of first-order equations:

Let:

$f1=f$

$f2=f'$

$f3=f''$

Then:

$f1'=f2$

$f2'=f3$

$f3'=-1/2\ f1\ f3$

The unknown initial condition is:

$f''(0) = a$

where a is determined using the shooting method.

The numerical solution is obtained using fourth-order Runge–Kutta integration until:

$f'(\eta) \to 1$

The calculated value is:

$f''(0) = 0.33206$

4. RESULTS AND DISCUSSION

4.1 Velocity Distribution

The dimensionless velocity profile $f'(\eta)$ increases continuously from zero at the plate surface and approaches unity away from the plate.

This indicates:

- No-slip condition at the wall.

- Increasing fluid velocity away from the surface.

- Formation of a viscous boundary layer.

4.2 Boundary Layer Thickness

The boundary layer thickness increases in the downstream direction because viscous effects diffuse further from the plate.

The thickness relationship is approximately:

$\delta/x \approx 5/\sqrt{Re_x}$

where Re_x is the local Reynolds number.

4.3 Skin Friction Coefficient

The wall shear stress is:

$\tau w = \mu(\partial u/\partial y)y=0$

Using the similarity solution:

$\tau w =$

$\mu U\infty\sqrt{(U\infty/\nu x)}f''(0)$

The local skin friction coefficient becomes:

$Cf = 0.664/Re_x\hat{}(1/2)$

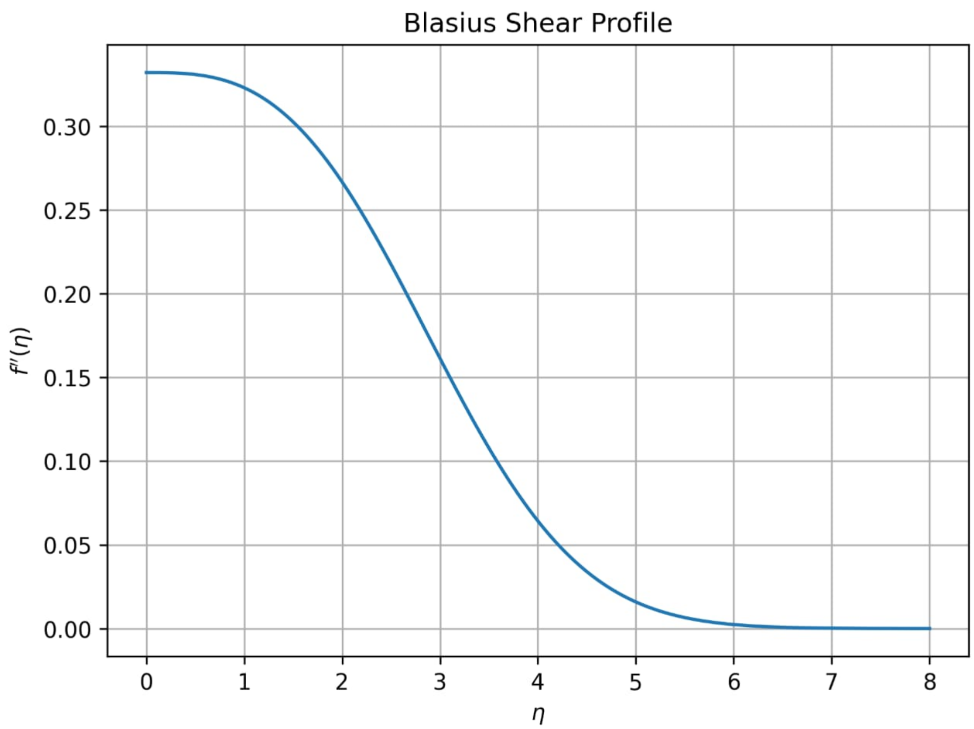

Figure1 This shows that skin friction decreases as Reynolds number increases.

4.4. ENGINEERING APPLICATIONS

The Blasius equation provides the basis for studying:

1. Aerodynamic drag on aircraft surfaces.
2. Heat transfer boundary layers.
3. Mass diffusion problems.
4. Magnetohydrodynamic boundary layers.
5. Non-Newtonian fluid flow over stretching surfaces.

Modern studies often modify the Blasius equation by adding effects such as magnetic field, thermal radiation, chemical reaction, and viscous dissipation.

5.0. CONCLUSION

In this Study a detailed analysis of the classical Blasius boundary layer equation has been presented. The governing Navier–Stokes equations were simplified using boundary layer approximations and transformed into a nonlinear third-order ordinary differential equation through similarity variables.

The numerical solution confirms the classical Blasius result:

$f''(0) = 0.33206$

The obtained velocity distribution and skin friction characteristics demonstrate the importance of similarity solutions in viscous flow analysis. The present work provides a foundation for future investigations involving advanced fluid models and coupled transport phenomena.

REFERENCES

Abbas, Z., Hayat, T., & Sajid, M. (2008). MHD boundary-layer flow of an upper-convected Maxwell fluid over a stretching sheet. Chaos, Solitons & Fractals, 38, 1245–1253.

Abbasbandy, S. (2007). A numerical solution of Blasius equation by Adomian decomposition method and comparison with homotopy perturbation method. Applied Mathematics and Computation, 186(1), 594–600.

Ahmad, S., & Khan, M. (2015). Analysis of boundary layer flow problems using similarity transformations. Journal of Applied Fluid Mechanics, 8, 1–10.

Anderson, J. D. (2017). Fundamentals of aerodynamics (6th ed.). McGraw-Hill Education.

Blasius, H. (1908). Grenzschichten in Flüssigkeiten mit kleiner Reibung. Zeitschrift für Mathematik und Physik, 56, 1–37.

Cebeci, T., & Cousteix, J. (2005). Modeling and computation of boundary-layer flows. Springer.

Cortell, R. (2005). Flow and heat transfer of a fluid through a porous medium over a stretching surface with internal heat generation. International Journal of Heat and Mass Transfer, 48, 3866–3876.

Crane, L. J. (1970). Flow past a stretching plate. Zeitschrift für Angewandte Mathematik und Physik, 21, 645–647.

Hayat, T., Abbas, Z., & Sajid, M. (2007). MHD boundary layer flow of an upper-convected Maxwell fluid over a porous stretching sheet. International Journal of Non-Linear Mechanics, 42, 1050–1058.

Jafar, K., & Nazar, R. (2009). Boundary layer flow and heat transfer over a stretching sheet with viscous dissipation. International Communications in Heat and Mass Transfer, 36, 333–337.

Keller, H. B. (1978). Numerical methods in boundary layer theory. Annual Review of Fluid Mechanics, 10, 417–433.

Kundu, P. K., Cohen, I. M., & Dowling, D. R. (2015). Fluid Mechanics.

Liao, S. J. (2003). Beyond perturbation: Introduction to the homotopy analysis method. Chapman & Hall/CRC.

Mukhopadhyay, S. (2013). Effects of thermal radiation and variable viscosity on stagnation point flow past a stretching sheet. International Journal of Heat and Mass Transfer, 57, 501–507.

Na, T. Y. (1979). Computational methods in engineering boundary value problems. Academic Press.

Prandtl, L. (1904). Über Flüssigkeitsbewegung bei sehr kleiner Reibung. In Proceedings of the Third International Mathematical Congress (Heidelberg).

Rosenhead, L. (1963). Laminar boundary layers. Oxford University Press.

Sakiadis, B. C. (1961). Boundary-layer behavior on continuous solid surfaces: I. Boundary-layer equations for two-dimensional and axisymmetric flow. AIChE Journal, 7(1), 26–28.

Sakiadis, B. C. (1961). Boundary-layer behavior on continuous solid surfaces: II. The boundary layer on a continuous flat surface. AIChE Journal, 7(2), 221–225.

Schlichting, H., & Gersten, K. (2017). Boundary-layer theory (9th ed.). Springer.

Schowalter, W. R. (1960). The application of boundary-layer theory to power-law pseudoplastic fluids. AIChE Journal, 6(1), 24–28.

Sparrow, E. M., & Gregg, J. L. (1958). Similar solutions for free convection from a nonisothermal vertical plate. Transactions of the ASME, 80, 379–386.

Wazwaz, A. M. (2009). Partial differential equations and solitary waves theory. Springer.

White, F. M. (2016). Fluid mechanics (8th ed.). McGraw-Hill Education.