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Computational and Statistical Study of the Relationship Between σ(n) and φ(n) in Natural Numbers

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Abstract

The focus of this work is to understand the classification of natural numbers, by implementing a computational approach using the Sum of Divisors Function by applying MATLAB. Numbers that are classified into three broad groups: Deficient Numbers, Perfect Numbers and Abundant Numbers based on the relationship between the sum of their Positive Divisors and twice the number. The first aim of this work is to break down the frequency of these classes into various groups in numerical ranges and to study the behaviour of these classes as the range increases. A sum of divisors algorithm in MATLAB was developed to compute the sum of divisors of each natural number and classify it as an either abundant or deficient number. Several runs totaling up to 100,000 were performed using the system. The outcome exhibited that the number class of Deficient numbers are found in larger numbers then the numbers that are Abundant, which is about 1-quarter of the numbers. The numbers that were found to be perfect were found to be very few, being only four in the range tested. The results reveal a statistically stable behavior of the ratio σ(n)/φ(n) across different numerical ranges, suggesting an underlying invariance in the interaction between divisor-based and multiplicative arithmetic functions.

Keywords

Number Theory,
Sum of Divisors Function,
Sigma Function σ(n),
Natural Numbers Classification,
Deficient Numbers

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1. Introduction

Number Theory is a very old and basic branch of Mathematics, which deals with the properties, relation and structure of integers. Although classical, it is still important enough in modern mathematics and computer science for applications like cryptography, coding theory, and computational mathematics [1].

The arithmetic function 'Sum of Divisors Function' of a number is one of such importance which forms a part of one of the very important concepts in Number Theory. This is a function that maps each natural number onto the sum of all the positive factors of it; gives insight into the interior structure of numbers [2]. This is a relatively simplest definition of a function, but its behaviour generates interesting and complex patterns which have been being studied since centuries [3].

Natural numbers can be categorized as follows: Deficient numbers, Perfect numbers and Abundant numbers based on Sum of Divisors Function. This classification depends on the comparison done between the sum of divisors of a number and 2 times the number. When the sum of the proper divisors of a number is less than 2 times the number, the number is defective, equal, the number is perfect, and > 2 times the number, the number is abundant. The use of this simple criterion yields a very interesting mathematical structure, and of course, many questions regarding the distribution and frequency of these classes within N arise [4].

In particular, have been studied extensively from ancient times because of their scarcity and special properties, perfect numbers. Very little is known about the distribution of the perfect numbers, as there are only a few such numbers known. Conversely, deficiencies and excesses occur with a noticeable frequency and have statistically predictable characteristics in long ranges of numbers [5].

New large-scale experiments investigating these mathematical concepts can now be conducted in the new large-scale experiments with the advancement of computational tools and environments like MATLAB. Recently, mathematical concepts can even be investigated experimentally on a large scale with the help of computational tools and programming environments like MATLAB. Rather than recourse to theoretical analysis, computational experiments can be used to study thousands or even millions of numbers efficiently, allowing a statistician to get an intrinsically visual and clear idea of the behavior of a mathematical experiment without having to examine it individually, to observe its behavior and begin to detect regularities within it.

This work uses a computational method to investigate the problem of the classification of natural numbers using the Sum of Divisors Function. A practical algorithm for computing divisor sums, classification of numbers and presenting the results statistically and graphically is implemented, using MATLAB. The primary objective is to combine theory and computation of these number classes in order to study the distribution, and confirm mathematical properties that were known by experimental data.

The addition of theory and computation not only aids in understanding classical mathematical concepts, but also illustrates the application of the power of modern computational tools in mathematical research and data analysis.

Besides divisor functions, the classification of natural numbers by them, there has been recently interest in the statistical behaviour of relations between the various divisor functions. One choice of functions made up of σ(n) and φ(n) helps provide a deeper understanding of the structure of integers of a particular kind particularly. While the classification system below is classical, this study explores the statistical stability of the ratio σ(n)/φ(n) in various ranges of the number.

2. Problem Statement

The division of natural numbers into Deficient numbers, Perfect numbers, and Abundant numbers are known and also well accepted in number theory, but only such classification does not give a clear idea about the relation between arithmetic functions from statistical point of view.

In particular, statistical and computational studies, for large ranges of numbers, on the interplay between the Sum of Divisors Function σ(n) and Euler's Totient Function φ(n) have not been intensive enough. The major problem to be solved in the current study is:
What is the statistical behavior of σ(n)/φ(n) for numbers n of various classes and is there any stability or structural invariance with respect to large intervals of n?.

3. Objectives of the Study

The main objectives of this research are:

1. Study the mathematical properties of the sum of divisors function σ(n) and Euler's function φ(n).
2. Classify natural numbers into imperfect, perfect, and abundant categories using σ(n).
3. Develop a computational model based on MATLAB to evaluate σ(n) and φ(n).
4. Analyze the statistical behavior of the ratio σ(n)/φ(n) across different ranges of natural numbers.
5. Verify the stability of the ratio σ(n)/φ(n) over different numerical intervals.
6. Compare the behavior of this ratio across different categories of natural numbers.
7. Present graphical and statistical representations of the results obtained.

4. Research Importance

This research is important because it uniquely integrates classical number theory with the computational and statistical analysis techniques. The study extends the classical framework by including a statistical investigation of the relationship between the Sum of Divisors Function σ(n) and Euler's totient function φ(n): It is well known that natural numbers can be classified according to the Sum of Divisors Function but in this study the study is extended to a statistical investigation of the relation between the Sum of Divisors Function σ(n) and Euler's Totient Function φ(n).

The importance of this study lies in the following aspects:

- It provides a deeper statistical understanding of the behavior of arithmetic functions over large sets of natural numbers.
- It highlights a previously underexplored relationship between σ(n) and φ(n) through the ratio σ(n)/φ(n).
- It demonstrates the effectiveness of MATLAB as a computational tool for

large-scale numerical experimentation in number theory.

- It bridges the gap between theoretical mathematics and experimental computational analysis.
- It reveals a statistically stable behavior of arithmetic functions across different numerical ranges, which may motivate further theoretical investigation.
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5. Scope of the Study

It is a very little study that goes only into computational aspects and statistical aspects of a natural number regarding the Sum of Divisors Function σ(n) and Euler's Totient Function φ(n). The research is based on the following borders:

- The study is limited to values in a finite range of natural numbers which can be as large as 100,000.
- Dividing numbers into equal three categories: Deficient, Perfect, Abundant numbers is done according to the classical definitions.
- The statistical analysis is based on the ratio σ(n)/φ(n) and its activity in different ranges of numbers
- The implementation is performed using MATLAB, without reliance on external symbolic computation software.
- The study does not aim to establish new theoretical proofs in number theory, but rather to provide computational evidence and statistical observations regarding known arithmetic functions.

6. Previous Studies

The authors present a family of related divisor functions which count the number of ordered factorizations of integers with certain properties. They created recurrence formulas and explicit formulas for these arithmetic functions and showed how they relate to the generalized divisor-counting methods. Besides, the study related these functions to the counting of sum systems in the form of expressions that count sum systems using divisor functions of these systems. This research gave an emphasis on the importance of the divisor functions in the combinatorial number theory and counting problems [6].

In this work, attention is paid to s-practical numbers which are the natural generalization of practical numbers by the sum-of-proper-divisors function. The authors looked into finding lower and upper bounds for the counting function of s-practical numbers. They also sorted some families of numbers, some of which were of the form s_practical with exactly two prime factors, and some of them were composite. Results lead to the understanding of the relationship between divisor functions and numbers classification [7].

The authors used techniques from the theory of modular forms to study convolution sums related to the sum-of-divisors function. Convolution sums were derived explicitly for some values of the parameters. It was then shown how to employ these formulas to take account of the number of positive integer representations of certain octonary quadratic forms. The technique of divisor functions was shown to be useful in higher level analytic and algebraic number theory [8].

In this study it was suggested that a generalized version of the sum-of-divisors function could be constructed while maintaining the multiplicative properties of the sum-of-divisors function. In the wake of this generalization the authors invented new classes of superabundant numbers and colossally abundant numbers. Aside from the sum-of-divisors functions, the study offered alternative ways using Euler's totient function to construct and analyze such special number classes. The results highlighted the importance of the divisor functions in the analysis of highly composite structures and related issues in the field of analytic number theory [9].

The extension of the divisors of natural numbers is explored and uncovering new relationships. The authors investigated techniques which can be used to help count divisors and to grasp the behaviour of prime numbers. The suggested relations are not primarily primality tests but rather were proposed as a means for investigating prime distributions and as a way to advance the research in the Riemann zeta function and its relationship to number theory [10].

Table 1 shows Comparison with Previous Studies.

Study	**Main Topic**	**Methodology**	**Main Contribution**
[6]	Associated divisor functions and ordered factorizations	Theoretical and combinatorial analysis	Derived recurrence relations and counting formulas for factorization structures
[7]	s-practical numbers	Analytical number theory	Investigated classification and counting of s-practical numbers
[8]	Convolution sums of divisor functions	Modular forms and analytical methods	Derived formulas for convolution sums and quadratic forms
[9]	Generalized sum-of-divisors functions	Theoretical analysis	Introduced generalized divisor functions and abundant number variants
[10]	Divisor relations and prime numbers	Number-theoretic analysis	Developed new divisor relations linked to prime number behavior
Present Study	Classification of natural numbers using the Sum of Divisors Function	MATLAB-based computational approach	Developed an algorithm to classify natural numbers into Deficient, Perfect, and Abundant numbers and analyzed their statistical distribution

Most of the literature found was about factorization counting, practical numbers, convolution sums, generalizations of divisor functions and divisor and prime number theories. The present study focusses on computational analysis of the Sum of Divisors Function in MATLAB in contrast. A proposed method is an amalgamation of the idea and the numerical experimentation to explore the distribution of Deficient, Perfect and Abundant numbers over a vast numeric range and present the outcome graphically represented statistically.

7. Methodology

The methodology of this study is based on a computational experimental approach. The steps are as follows:

1. Define a numerical range of natural numbers for analysis.
2. Compute the Sum of Divisors Function for each number in the range.
3. Compare the result with twice the value of the number.
4. Classify each number into one of the three categories:
    - Deficient if the sum is less than 2n.
    - Perfect if the sum is equal to 2n.
    - Abundant if the sum is greater than 2n.
5. Use MATLAB to implement this algorithm.
6. Compile statistical data and create graphing presentations for analysis.
7. To note the patterns and trends in data across a range of numbers.

8. Proposed System

8.1 Introduction

The proposed system aims to develop a classification scheme for natural numbers based on the sum of divisors function, using MATLAB software. The system provides a simple and automated method for analyzing natural numbers within various numerical ranges. Sorting is achieved by calculating the sum of all positive divisors of a given number and comparing the result to twice its value to determine its category.

8.2 Structure of the Proposed System

The proposed system has four stages, they are:

1. Setting up the range of numbers to be studied.
2. Calculating the Sum of Divisors Function for each natural number within the specified range.
3. Appliance of the rules of classification.
4. Organising the answers as tables of statistics and as graphs.

The classification is done using the Sum of Divisors Function, which is the sum of all the positive divisors of a positive natural number. Once the value of each function is calculated each number is determined by the following rules:

If the sum of the Numbers which divide a number is less than twice that number, it is known as a Deficient number.

A Perfect Number is said to be a number whose sum of the divisors is twice the number.

A number is called Abundant Number, if the total of its divisors exceeds twice the number.

8.3 Algorithm Description

An algorithm was developed using MATLAB to automatically classify natural numbers. The algorithm iterates through all numbers within a defined range. For each number, all positive divisors are identified and summed. The resulting value is then compared to twice the number to determine its classification class.

After processing all the numbers, the algorithm calculates the total number of deficient, whole, and abundant numbers, and produces statistical results and graphs for further analysis.

8.4 MATLAB Implementation

MATLAB is used to implement this proposed system as it has a capability of powerful numerical computations and visualization tools. The program is designed to analyze numerical ranges automatically multiple. The following ranges were explored:

- 1 to 1,000
- 1 to 10,000
- 1 to 50,000
- 1 to 100,000

The processed data were presented, displayed and visualized by various graphics, to ease understanding and comparing.

9. Results and Discussion

9.1 Results

Various numerical ranges were selected for the proposed system. The results of the classification performed are presented in Table 2.

Table 2 Classification Results

Range	Deficient Numbers	Perfect Numbers	Abundant Numbers
1–1,000	751	3	246
1–10,000	7,508	4	2,488
1–50,000	37,602	4	12,394
1–100,000	75,201	4	24,795

Here are the Perfect Numbers found in the range of numbers studied: 6, 28, 496, and 8128

The obtained values are the same as the known values of perfect numbers less than 100,000, which shows correctness of the algorithm implemented.

It is clear that the majority of the data (N=10000; Form the overwhelming majority) dominates – there are more than 75,000 entries (about 7.5 x 104) - see figure 1..

Abundant Numbers: Rank second, total frequency is ~25,000 (~2.5 x 104).

Perfect Numbers: Due to their extreme rarity, their corresponding bar is practically invisible on this scale, highlighting their scarcity in number theory.

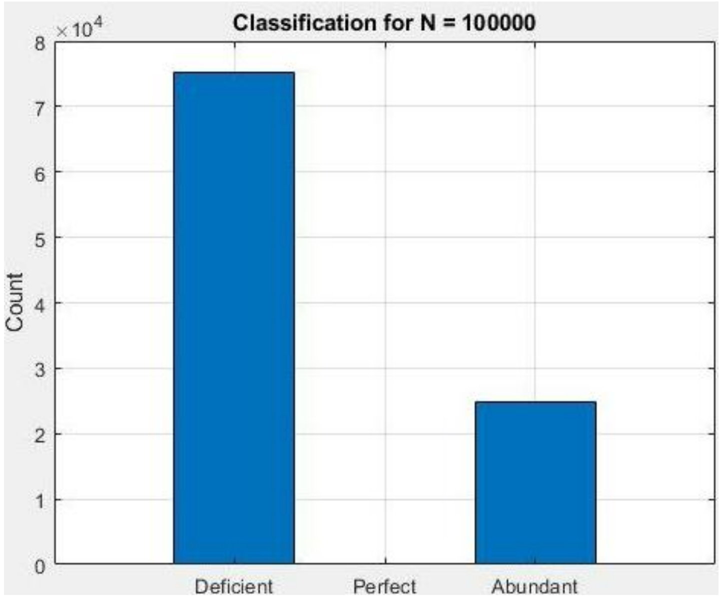

Figure 1 classification of N=10000

The relative distribution of the N=100,000 is shown in Figure 2: Deficient numbers" make up the biggest population, 75% of the total. Abundant numbers: These are the greatest in numbers, about 25%. These are a very small number of the numbers (less than 1% (<1%)). It is no wonder that their graph has no frequency, and confirms that perfect numbers are very scarce.

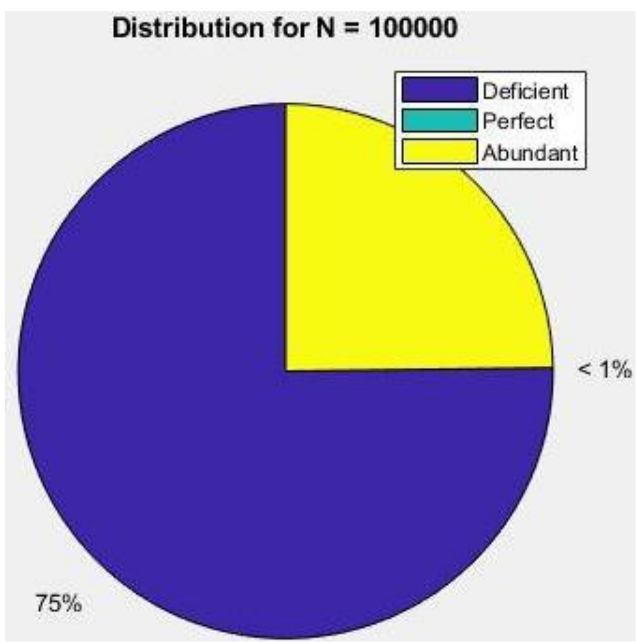

Figure 2 Distribution of N=100000

With growing N the number classes consist of the deficient numbers (blue) and the abundant numbers (orange), as illustrated in figure 3, both classes have a linear growth with a constant ratio. This means that the density of such numbers is statistically stationary for long periods of time.

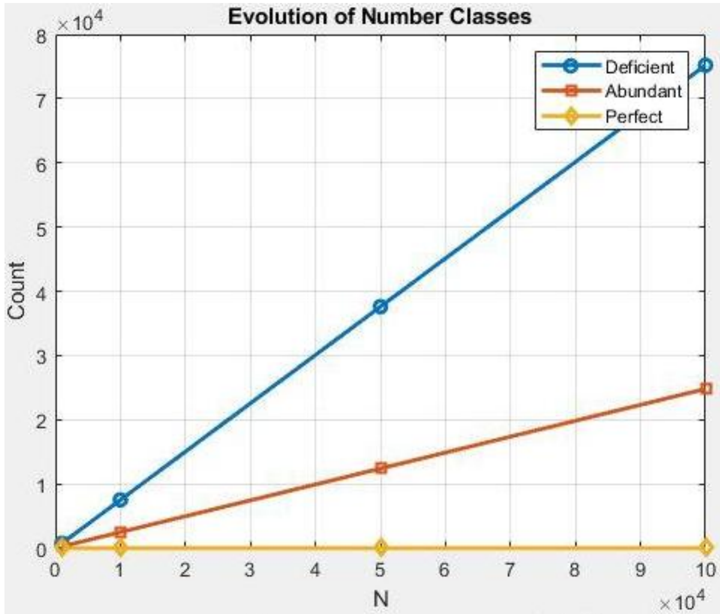

Figure 3 Evolution of number classes

Statistical Stability of the Ratio

The numerical experiments reveal that the mean value of σ(n)/φ(n) is more or less steady for various values of N. In the case of Deficient numbers, the mean value level off around 2.33 and for Perfect numbers the average value remains close to 4.7 and for Abundant numbers around 7.5.

The consistency suggests that the ratio has a stable statistical structure, meaning that it is not significantly affected by the number of data points. This indicates that the Sum of Divisors Function is statistically similar to Euler's Totient Function if the range of natural numbers ({N}) is extended.

Figure 4 above shows the mean divisor/euler's totient function Mean sigma(n)/phi(n) varying with sample size N under the assumption that these functions take integer values. The horizontal stabilization of the mean ratio for the three number classes is excellent, starting from a very small adjustment at the beginning, as N increases by 105.The mean ratio value becomes stabilized horizontally for all three numbers classes after a small adjustment at the start as N becomes larger of 105. Distinct Class Stratification: There is good distinct hierarchical split which targets the highest mean ratio, which stabilizes within the range of c.a. 7.5, which represent Abundant Numbers. For Perfect Numbers fill the middle line - become stable around the value of 4.7. Lowest numbers – Have the lowest and most constricted ratio, around 2.3.

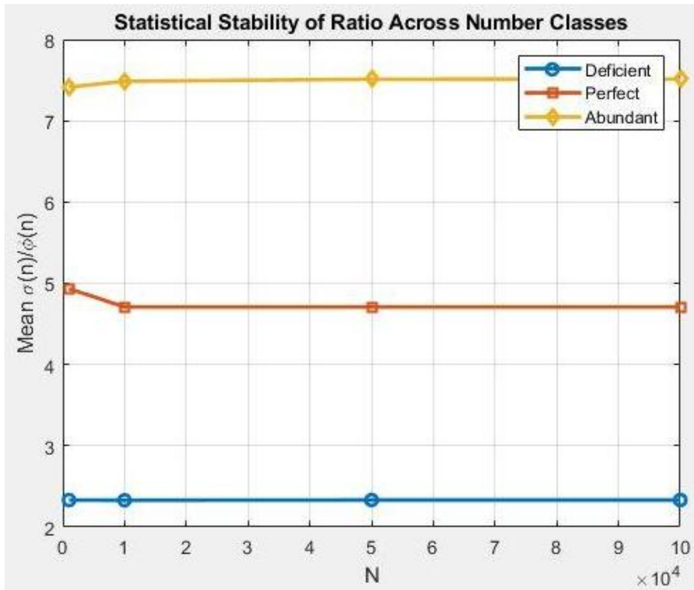

Figure 4 Statistical Stability

Figure 5 illustrates the absolute frequency (count) of each number class across different upper bounds of N.

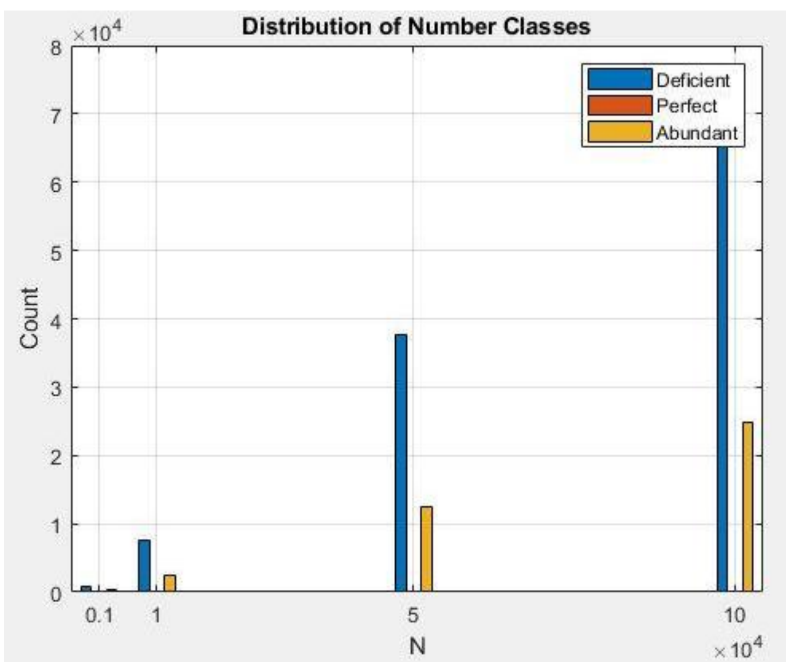

Figure 5 distribution of number of classes

9.2 Discussion

The results obtained provide a deeper understanding of the relationship between divisor-based arithmetic functions and multiplicative arithmetic functions. While the classification of numbers as deficient, perfect, and abundant is well-established, this study offers a statistical perspective based on the ratio $\sigma(n)/\varphi(n)$.

A key observation is the stability of the average values of this ratio across different numerical ranges. This indicates that the ratio's behavior is not affected by scale; that is, increasing the upper limit of the natural numbers does not significantly alter its statistical properties within each category.

Furthermore, abundant numbers consistently exhibit higher values for the ratio compared to deficient numbers, reflecting the richness of their divisor structure. Perfect numbers, despite their rarity, occupy an intermediate position in the distribution, consistent with the balanced properties of their divisors.

These results highlight a statistical regularity that has not been adequately explored previously in the interaction between $\sigma(n)$ and $\varphi(n)$, which may open new horizons for further research in analytic and arithmetic number theory.

9.3 Evaluation of the Proposed System

The proposed system is successful in categorizing numbers which are natural numbers based on a Sum of Divisors Function. The results obtained were completely in agreement with the theoretical and the known mathematical properties of these functions. Additionally, the MATLAB environment was found to be a successful platform to program numerical algorithms and produce graphical and statistical analysis.

The system has used satisfactory performance in test range of numbers and yields accurate classification between test sets which can be used in further research work in Number theory.

10. Conclusion

This research examined categorizing natural numbers according to the Sum of Divisors Function by using computational method that has been carried out with MATLAB. The primary goal was to examine the behavior and distribution of the natural numbers in each of the categories of Deficient, Perfect and Abundant numbers.

The algorithm produced in MATLAB was effective in obtaining the sum of the divisors of each of the natural numbers between the specified ranges and also categorizing the numbers based on the specified mathematical property. Experimental tests carried out for a range up to 100,000 yielded consistencies as well as a stable pattern for all ranges. The answer was that there are a lot of Deficient numbers in the range of testable integers, that is, about 75% of the integers in these testable ranges. There were 4 times that were Perfect numbers, only a few of which were discovered in the range up to 100,000, and approximately 1/4 of the data had abundant numbers.

As these observations are in tune with the well-known theoretical results in Number Theory they attest a behaviour of distribution in these number classes as was anticipated. Furthermore, the paper shows how computer packages such as MATLAB can be used to analyse the mathematical structures and generate visual and statistical data that is not readily attainable via theoretical methods. Numerical experiments and mathematical theory helped to make it clear what the divisor-based classifications were. The rest of the paper elaborates the proposed system which combines the theoretical elements of Number Theory and computational analysis, highlighting it as a practical means to explore extensive sets of natural numbers and their properties.

11. Future Work

Better implementations would be needed if the Sum of Divisors Function (SDF) computation technique is used instead, as it would rely on prime factorization and other techniques which would significantly decrease the amount of computation.

The research can be extended futher for others special types of numbers of which Amicable Numbers, Sociable Numbers and highly Composite Numbers.

This observation could be analysed further to provide some description of the stability of the ratio of Deficient numbers to Abundant numbers.

Incorporating graph and machine learning into a single application. There are patterns that exist in the classifications of numbers, but which do not seem to be immediately apparent, but which can be recognized using techniques of advanced visualization or machine learning models.

These extensions will, in general, help to justify further applications of these into Number Theory and further build understanding of divisor-based functions.

The study also points to a stable behavior of the ratio $\sigma(n)/\varphi(n)$ that may be taken as evidence of potential invariance properties of interactions between arithmetic functions.

References

[1] Nagell, Trygve. Introduction to number theory. Vol. 163. American Mathematical Soc., 2021.

[2] Hasse, Helmut. Number theory. Walter de Gruyter GmbH & Co KG, 2022.

[3] Iwaniec, Henryk, and Emmanuel Kowalski. Analytic number theory. Vol. 53. American Mathematical Soc., 2021.

[4] Nicolas, Jean-Louis. "The sum of divisors function and the Riemann hypothesis." The Ramanujan Journal 58.4 (2022): 1113-1157.

[5] Vega, Frank. "Note for the Perfect Numbers." ScienceOpen Preprints (2024).

[6] Lettington, Matthew C., and Karl Michael Schmidt. "Divisor functions and the number of sum systems." arXiv preprint arXiv:1910.02455 (2019).

[7] Reuijl, Anniek. Representing integers with sums of proper divisors. MS thesis. 2026.

[8] Ntienjem, Ebénézer. "Evaluation of the convolution sum involving the sum of divisors function for 22, 44 and 52" Open Mathematics, vol. 15, no. 1, 2017, pp. 446-458. https://doi.org/10.1515/math-2017-0041

[9] Cox, Darrell, Sourangshu Ghosh, and Eldar Sultanow. "A Generalization of the Sum of Divisors Function." (2022).

[10] da Silva, Hamilton Brito. "New properties of divisors of natural number." (2023).