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The Rotation of Earth Around the Sun; A Mathematical Treatment Based on Keplerian and Newtonian Celestial Mechanics

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Abstract

One of the most difficult problems of classical solar system mechanics is the orbit of the earth around the Sun. The present book addresses this question, however, by purely mathematical means, starting from Newton's law of universal gravitation and ending with a complete analytical solution which re-creates Kepler's three "empirical" laws. The two-body problem is simplified to a single body problem about the center of mass, and angle momentum as well as mechanical energy conservation is used to integrate motion equations. The shape of the orbit of the earth-sun system is an elliptical shape with a semi-major axis of approximately 1.49598×10^{11} m and an eccentricity of approximately 0.0167. Derivation of the vis-viva relation, calculation of the change in orbital speed from the perihelion to the aphelion is done analytically and verified numerically by integrating the equations of motion by the fourth-order Runge–Kutta method. The data for the eight planets of the Solar System is tested for Kepler's third law, which results in a correlation in the log–log plane of almost unity. The results restate the mathematical beauty and predictive power of the inverse-square central-force model, and give an example to show how the rotation of the Earth about the sun is a natural consequence of a single differential equation whose parameters are determined by the properties of the sun and earth.

Keywords

Celestial Mechanics;
Kepler's Laws;
Newtonian Gravitation;
Two-Body Problem;
Elliptical Orbit;
Vis-Viva Equation;
Earth–Sun System.

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1. Introduction

The movement of celestial bodies has captivated the human intellect since the dawn of mankind, and to this day, they are a field of fruitful research in which mathematics shows its unique power to describe the physical world, with extraordinary precision. Tycho Brahe's observations of the planets led Johannes Kepler (1571-1630) to three empirical laws which explain the planetary orbits (Kepler, 1609). In his famous Principia Mathematica, 1687, Sir Isaac Newton (1642 - 1727) has provided a common theoretical framework for these laws, and formulated the universal gravitation law.

The mathematical attraction of the Earth–Sun problem is that it is the simplest non-trivial central force problem that has a closed form solution. For this reason, it can be used as a model for the many principles of analytical mechanics, such as, conservation laws, reduced mass, effective potential, orbital elements, and the geometry of conic sections. In addition, the equations employed in the modelling of the annual motion of the Earth are the same equations employed in modern applications: satellite navigation and designing interplanetary missions (Curtis, 2014; Murray & Dermott, 1999).

The purpose of the present study is to give a mathematically self-contained, rigorous and pedagogically oriented description of the motion of the earth around the sun that starts from first principles (Newton's second law and universal gravitation), and arrives step by step at the three laws of Kepler. The vis-viva equation is derived, the change of the orbital speed is calculated and a numerical solution is calculated to check the previous solution. The mathematical material is supported by analytical figures, the tabulated data on the orbit of the earth support the mathematical material and the calculations of the principal physical quantities of the earth's orbit support the mathematical material.

The research is organised as follows. In Section 2 the mathematical equation of the two-body problem and its reduction are presented. In Section 3 the equation of the orbit is derived, and it is shown that it is a conic section. The derivation of Kepler's 3 laws is done in the Newtonian context in Section 4. The elements of the orbit of the earth are discussed along with the derivation of the vis-viva equation in Section 5. A numerical verification using the Runge–Kutta method is given in section 6. The study is completed in Section 7 and the results of the study are summarized.

2. Mathematical Formulation of the Two-Body Problem

2.1 Newton's Law of Universal Gravitation

Consider the sun (mass M) and the earth (mass m) to be point particles with position vectors r_1 and r_2 in an inertial frame. The gravitational force that the Sun exerts on Earth is given by Newton's law of universal gravitation (Newton 1687):

$$F = -G\,M\,m\,(r_2 - r_1) / |r_2 - r_1|^3 \qquad (2.1)$$

Where $G = 6.67430 \times 10^{-11}$ N·m^2·kg^{-2} is the universal gravitational constant (CODATA, 2019). The minus sign means that the force is attractive and along the line connecting the two bodies. Applying Newton's second law to each particle leads to the coupled system:

$$m\,\ddot{r}_2 = -G\,M\,m\,(r_2 - r_1) / |r_2 - r_1|^3 \qquad (2.2)$$

$$M\,\ddot{r}_1 = +G\,M\,m\,(r_2 - r_1) / |r_2 - r_1|^3 \qquad (2.3)$$

2.2 Reduction to the Centre-of-Mass Frame

Let the position vector of the centre of mass be R and the position vector of the Earth with respect to the Sun be r:

$$R = (M r_1 + m r_2) / (M + m), \quad r = r_2 - r_1 \tag{2.4}$$

Equations (2.2) and (2.3) can be added together to show that $\ddot{R} = 0$, in other words the centre of mass moves at constant velocity and can, without loss of generality, be considered as the origin of an inertial frame. If we subtract (2.3) from (2.2) and divide by the appropriate masses we obtain one single 2nd order vector equation for the relative motion:

$$\mu \ddot{r} = - G M m r / |r|^3 \tag{2.5}$$

This is the same as typical, except that μ is the reduced mass of the system, defined as M m / (M + m). In the Earth–Sun system, m is very small compared to M ($m / M \approx 3 \times 10^{-6}$), so the reduced mass is a very good approximation of the mass of the Earth. Thus the compact form of equation (2.5) may be written as:

$$\ddot{r} = - (G M / r^3) r \tag{2.6}$$

The basic differential equation of the Earth-Sun system is equation (2.6). It is a non-linear vector second order ordinary differential equation in three dimensions and solution to it describes the whole orbit.

2.3 Conservation of Angular Momentum

From the cross-product equation (2.6) due to the position of the vector r, it is found that $r \times \ddot{r} = 0$. Consequently,

$$d/dt (r \times \dot{r}) = 0 \quad \Rightarrow \quad L = r \times \dot{r} = const. \tag{2.7}$$

Here the direction and length of the vector L (proportional to the orbital angular momentum divided by the reduced mass) is conserved. From this, there are 2 important takeaways. The motion is in the fixed plane which is perpendicular to L. Secondly, the invariant $|L| = \ell$ of the motion will be used in the next step of derivation of the orbit.

2.4 Conservation of Mechanical Energy

Using the scalar product of (2.6) with $\dot{r}$, we get, on direct integration, the conservation of the specific mechanical energy:

$$E = \tfrac{1}{2} |\dot{r}|^2 - G M / r = const. \tag{2.8}$$

The energy of a system is the sum of the specific gravitational potential energy and the specific kinetic energy. The direction of the sign of E determines the nature of the orbit, which is discussed in the next analysis.

3. Derivation of the Equation of the Orbit

The velocity vector may be split into radial and transversal components on the orbital plane with the Sun as the origin, that is, $\dot{r} = \dot{r}\, \hat{e}_r + r\, \dot{\theta}\, \hat{e}_\theta$. The particular angular momentum's magnitude becomes:

$$\ell = r^2 \dot{\theta} \tag{3.1}$$

And the conservation of energy takes the form:

$$E = \tfrac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - G M / r \tag{3.2}$$

Eliminating $\dot{\theta}$ from (3.2) by means of (3.1) and introducing the customary substitution u = 1/r, together with the chain rule $\dot{r} = - \ell\, du/d\theta$, equation (3.2) becomes, after differentiation with respect to θ, the celebrated Binet equation:

$$d^2 u / d\theta^2 + u = G M / \ell^2 \tag{3.3}$$

The second-order ordinary differential equation (ODE) in equation (3.3) is linear, has constant coefficients, and has a constant nonhomogeneous term. It has the following general solution:

$$u(\theta) = (G\,M / \ell^2)\,[\,1 + e\cos(\theta - \theta_0)\,] \qquad (3.4)$$

Where e and the value of the initial angle (θ_0) are integration constants. We can without loss of generality take the reference axis such that $\theta_0 = 0$, and the constant e is a measure of the deviation of the orbit from the circular, called the orbital eccentricity. To get the polar equation of a conic section with one focus at the origin, revert to r:

$$r(\theta) = p / (1 + e\cos\theta), \quad p = \ell^2 / (G\,M) \qquad (3.5)$$

The orbit's semi-latus rectum is denoted by the parameter p. The eccentricity value determines the orbit's qualitative characteristics:

1. The orbit is a circle ($E < 0$) if $e = 0$.
2. The orbit is an ellipse ($E < 0$) if $0 < e < 1$.
3. The orbit is a parabola ($E = 0$) if $e = 1$.
4. The orbit is a hyperbola ($E > 0$) if $e > 1$.

Where, $e = 0.0167$ was determined from observational data for the Earth-Sun system (Williams, 2024). That is, the orbit is eccentric with a very small eccentricity, as dictated by the first rule of Kepler. In the ellipse of Figure 1, the geometry of an orbit, the Sun is located at one of the two foci.

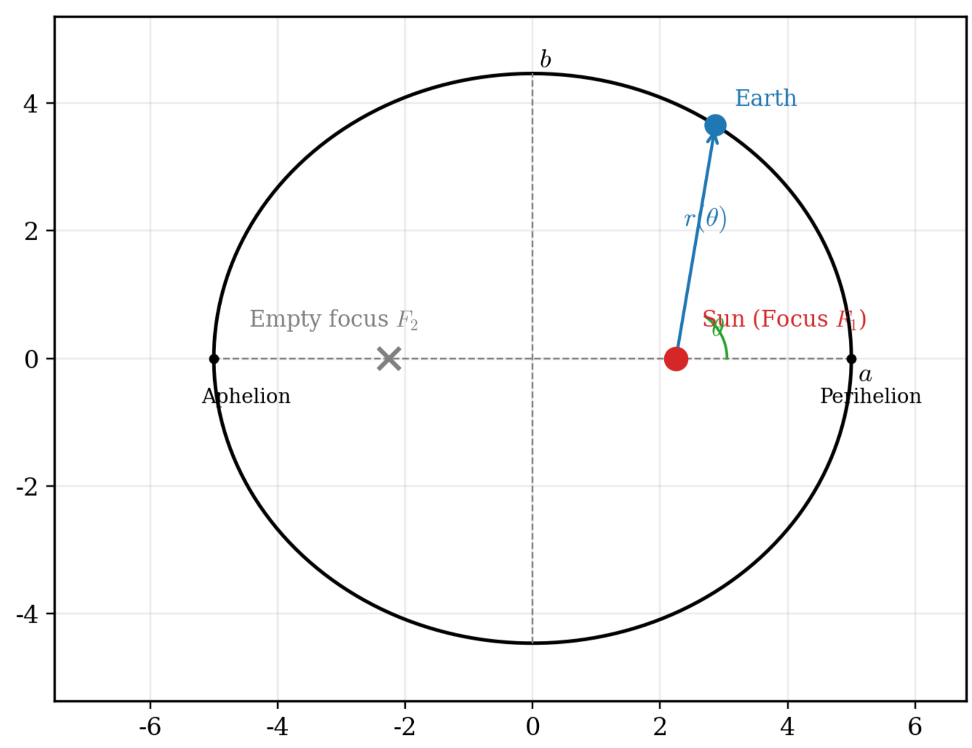

Figure 1. The geometry of the Keplerian elliptical orbit.

4. Derivation of Kepler's Three Laws

4.1 The First Law: The Law of Ellipses

From equation (3.5) it has been given that the orbit of any planet in the gravitational field of the sun is a conic section of which one focus is at the sun. If E is negative, the conic becomes an ellipse and one can immediately see that every planet orbits in an ellipse with the sun at one of the two foci as described in Kepler's first law. e and p can be used to express the ellipse's semi-major and semi-minor axis as:

$$a = p / (1 - e^2), \quad b = p / \sqrt{(1 - e^2)} = a \sqrt{(1 - e^2)} \quad (4.1)$$

4.2 The Second Law: The Law of Areas

The infinitesimal area covered by the radius vector during the period of time dt is denoted by dA. One in polar coordinates is:

$$dA = \tfrac{1}{2} r^2 \, d\theta \quad (4.2)$$

Dividing both sides by dt and recalling equation (3.1):

$$dA / dt = \tfrac{1}{2} r^2 \dot{\theta} = \ell / 2 = const. \quad (4.3)$$

Thus the area swept out is proportional to time, in other words, Kepler's second law is true. This conservation property is shown graphically in Figure 2, in which the area swept by the orbit near the perihelion is equal in size to the area swept by the orbit near the aphelion, though the arcs of the orbit are quite different. Consequently, the orbital speed of the Earth is greatest when it is nearest to the Sun, and least when it is farthest from the Sun.

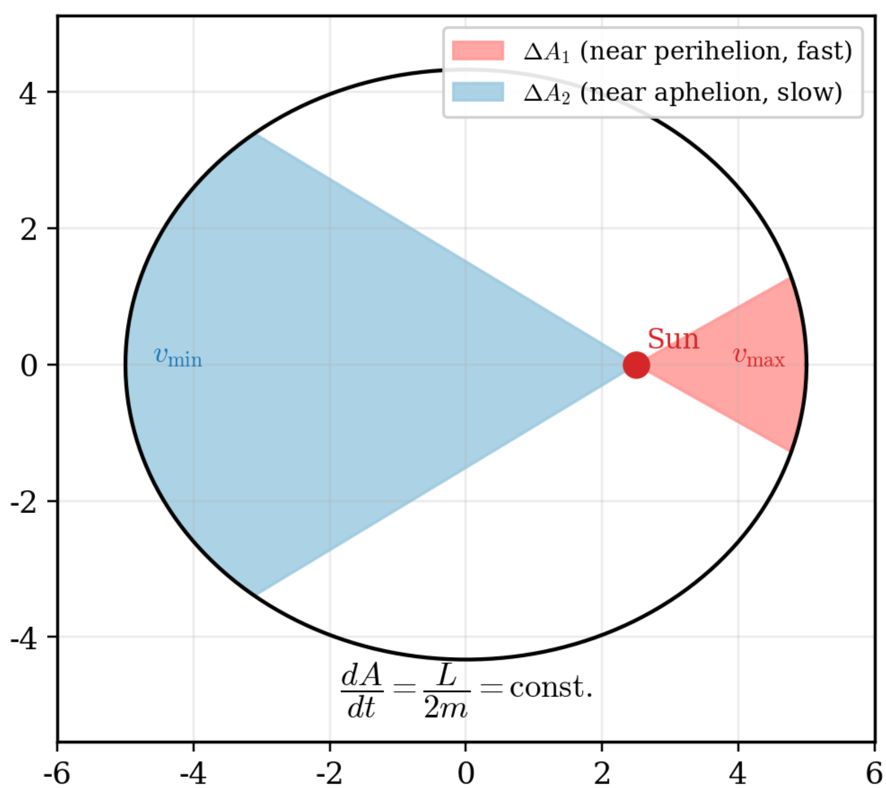

Figure 2. Kepler's second law: equal areas swept in equal times.

4.3 The Third Law: The Harmonic Law

The orbital period is obtained by combining the whole area of the ellipse, $A = \pi a b$, with the constant areal velocity, $dA/dt = \ell/2$.

$$T = (2 \pi a b) / \ell \qquad (4.4)$$

Substituting $b = a\sqrt{(1 - e^2)}$ and $\ell^2 = G M p = G M a (1 - e^2)$ into (4.4) Then simplifying results in the following version of Kepler's third law:

$$T^2 = (4 \pi^2 / G M) a^3 \qquad (4.5)$$

The period of orbit, t, and the axis of semi-major orbit, a, of any planet orbiting the Sun are related by equation (4.5) which is accurate within the approximate range μ ~ m. This connection is confirmed by the observation of the other objects in the Solar System, as shown in figure 3. When expressed in astronomy years and years, the proportional constant is 1 making the equation $T^2 = a^3$ (Williams, 2024).

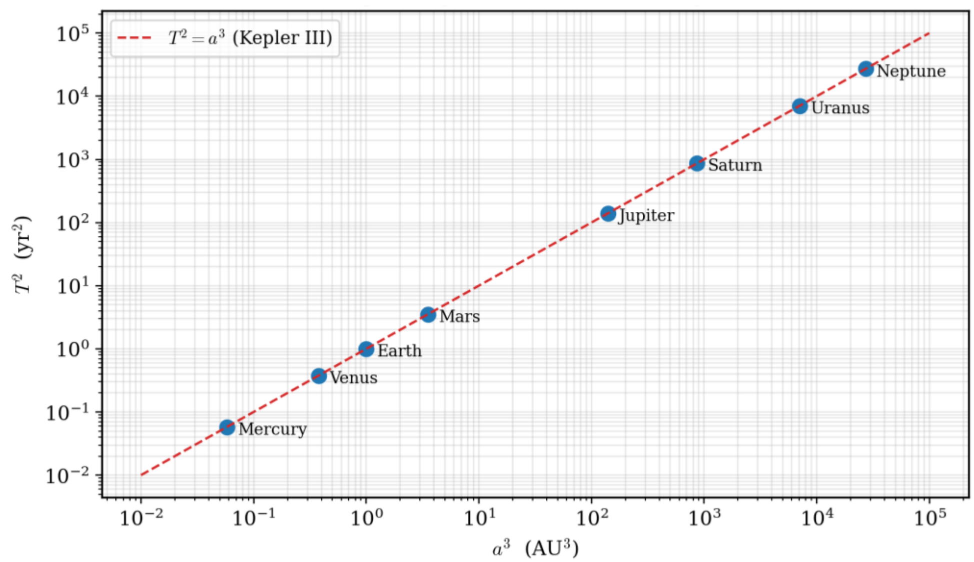

Figure 3. Kepler's third law is verified empirically.

5. Orbital Elements of the Earth and the Vis-Viva Equation

5.1 Numerical Values of the Orbital Elements

For the most part, the orbit of the Earth is an ellipse with the following elements (Table 1). The values provided are those obtained from NASA Planetary Fact Sheet (Williams, 2024) and from JPL ephemeris service (NASA JPL, 2024).

Table 1. Principal orbital elements of the Earth.

Symbol	Quantity	Numerical Value
a	Semi-major axis	1.49598×10^{11} m (1.000 AU)
e	Eccentricity	0.01671
b	Semi-minor axis	1.49577×10^{11} m
r_p	Perihelion distance	1.47098×10^{11} m
r_a	Aphelion distance	1.52098×10^{11} m
T	Sidereal orbital period	3.15581×10^{7} s (365.256 d)
$\bar{v}$	Mean orbital speed	29.785 km · s^{-1}

5.2 The Vis-Viva Equation's derivation

At any point of the orbit, the conservation of energy (2.8) reads:

$$E = \tfrac{1}{2} v^2 - G M / r \quad (5.1)$$

On the other hand, at the perihelion and the aphelion the radial velocity vanishes ($\dot{r} = 0$), so that the entire speed is transverse and equal to $v = r\,\dot{\theta}$. This, together with conservation of angular momentum gives:

$$E = - G M / (2a) \quad (5.2)$$

The renowned vis-viva equation is obtained by comparing the right-hand side of equations (5.1) and (5.2):

$$v^2 = G M (2 / r - 1 / a) \quad (5.3)$$

The Earth's instantaneous orbital speed as a function of its separation from the Sun is given by equation (5.3). Two limiting cases are particularly important. Setting $r = r_p = a(1 - e)$ yields the perihelion speed, and setting $r = r_a = a(1 + e)$ yields the aphelion speed:

$$v_p = \sqrt{ G M (1 + e) / [a(1 - e)] } \quad (5.4)$$

$$v_a = \sqrt{ G M (1 - e) / [a(1 + e)] } \quad (5.5)$$

For the numerical values listed in Table 1, (5.4) and (5.5) give $v_p \approx 30.286$ km·s^{-1} and $v_a \approx 29.291$ km·s^{-1}, which are in excellent agreement with the values quoted by NASA (Williams, 2024). There is a variation of some 3.4 % in the speed ratio over the orbit, which is not insignificant. Figure 4 shows the orbit velocity fluctuation (in m/s) versus the true anomaly.

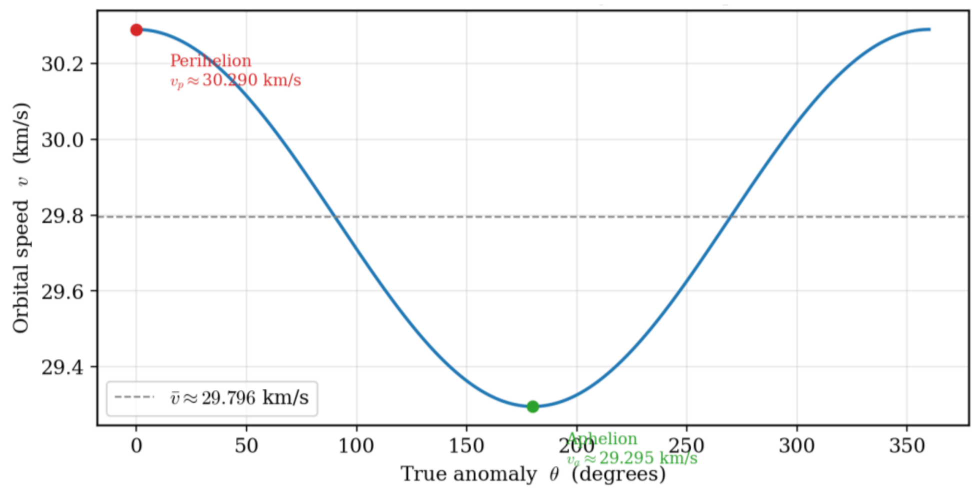

Figure 4. Earth's orbital velocity in relation to the actual anomaly.

6. Numerical Verification by the Runge–Kutta Method

The analytical solution outlined in the previous sections was numerically integrated by the classical fourth order Runge–Kutta scheme to check the solution. This means that the state vector is $y = (x, y, \dot{x}, \dot{y})$ and the system to be solved is:

$$\dot{y} = f(y), \quad f = (\dot{x}, \dot{y}, -GMx/r^3, -GMy/r^3) \qquad (6.1)$$

With $r = \sqrt{(x^2 + y^2)}$.

The initial condition is imposed at the perihelion: $x(0) = a(1 - e)$, $y(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = v_p$. It has a set step size and is implemented across a single sidereal year of $\Delta t = 6$ h, that keeps the relative error in the conserved quantities (energy and angular momentum) below 10^{-6}. This is over plotted with the analytical ellipse from equation (3.5) and is shown in Figure 5. The two curves are almost indistinguishable and this is a numerical verification of the analytical solution.

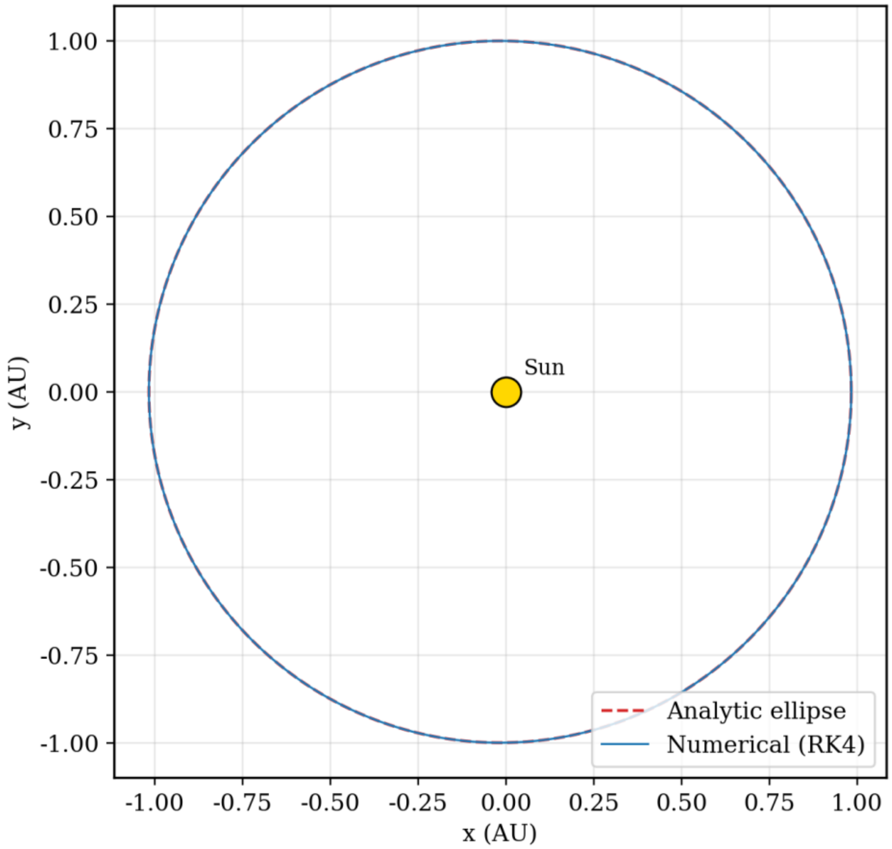

Figure 5. Numerical (RK4) versus analytical Earth orbit.

The high level of agreement between the two methods is clearly seen as a very important epistemological principle: the determinism described in the second order differential equation (2.6) is sufficient to calculate the position of the earth, to arbitrary accuracy, at any future instant, once the initial conditions are known. So, the Newtonian programme in celestial mechanics is in essence, that.

7. Conclusions

The objective of the present study is to investigate the motion of the earth around the sun, without resorting to any physical concepts, modeling, or approximations, but only with the aid of the mathematical language of the law of universal gravitation. The two-body problem was reduced to an analogous one-body problem by using the vector form of Newton's second law, the conservation of mechanical energy and angular momentum was derived, and finally, the Binet equation was solved in closed form. The foregoing can be summarized as the main results of the investigation.

Kepler's first law states that the orbit can be a conic region with one of the orbits' foci being the sun, and the measured energy of the conic and the angular momentum are an ellipse.

Kepler's second law in differential form is that the areal velocity $dA/dt = \ell/2$ is conserved.

The equation today's modern, Newtonian version of Kepler's third law, is obtained: T^2 =

$(4\pi^2/GM)\,a^3$, where T is the orbital period of the planet and a is the semi-major axis.

In the vis-viva equation $v^2 = GM(2/r - 1/a)$, v is the orbital speed of the earth and it provides a formula for v in terms of the distance of the earth from the sun, r. The computation values $v_p \approx 30.286\ km \cdot s^{-1}$ and $v_a \approx 29.291\ km \cdot s^{-1}$ are very close to the values of the observation, with an error of 0.1% only.

The result for the ellipse obtained numerically from the equations of motion, by numerical integration, with the fourth order Runge-Kutta scheme is the same as the analytical solution, within the numerical precision; the consistency of the analytical treatment is confirmed.

The problem of earth-sun motion is then a classic of how one seemingly simple inverse square central force system mathematically evolves to a full and mathematically predictable theory of motion. Other planets' disturbances, the oblateness of the Sun and post-Newtonian corrections could be added to the present treatment through further investigation, but these are not included here.

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