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Operational-Matrix Methods for Caputo Fractional Delay Differential Equations; A Critical Comparison of Shifted Third-Kind Chebyshev Polynomials and Genocchi Wavelets; As Review Article

Mohammed Saleh Hadi

Mathematical Department, Open Education College, Wasit, Iraq.

Abstract

Fractional delay differential equations (FDDEs) are a type of nonlocal memory fractional differential equations that have an explicit dependence on past states. This mixup is a source of confusion in the analysis and can make numerical solutions sensitive to the regularity of the solution, the delay geometry, the history data, and the conditioning of the basis chosen. There are reasons to trust operational-matrix methods, since they are based on a finite algebraic system instead of the governing equation, but comparisons published in the literature are sometimes made from different test problems, of different fractional order, of different truncation size, of different hardware, and of different error definitions. Such comparison does not demonstrate one basis is always more accurate and/or faster than another. This article reviews critically two representative methods: the shifted third-kind Chebyshev polynomial method, and the Genocchi wavelet method. A current mathematical framework is employed to allow identification of exact identities from projected and pseudo operational approximations. The analysis reveals that the shifted third kind Chebyshev polynomials are an orthogonal, globally supported set which is well adapted to smooth solutions and simple delay maps. While the Genocchi wavelets have compact support within dyadic subintervals and offer explicit operator constructions for a number of constant and proportional delays, the underlying Genocchi polynomials are not orthogonal, there is only approximate representation of the Caputo operator, and locality is restricted by the regularity and alignment of solution features. There is no general consensus in the literature about the relative ranking of the two methods due to the fact that no head-to-head benchmark was found that uses the same equations, reference solutions, tolerances, and computational environments. The review introduces a new selection framework based on evidence and a benchmark protocol for selecting a procedure that is reproducible. The conclusions that are reached are tentative, clear and appropriate for future comparative studies.

Keywords

Caputo Derivative;
Fractional Delay Differential Equation;
Operational Matrix;
Spectral Collocation;
Third-Kind Chebyshev Polynomials;
Genocchi Wavelets;
Numerical Reproducibility.

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Faculty of Science, ATBU Bauchi

Corresponding Author: Mohammed Saleh Hadi, Mathematical Department, Open Education College, Wasit, Iraq. Email: mohammedhadi@ouwasit.edu.iq

1. Introduction

If the current rate of change is influenced by the past history, then the differential equation is called a fractional differential equation. The Caputo formulation is based on a weekly singular convolution of an integer-order derivative with a kernel of the power-law type [1,2]. In the case of delay differential equations, there is another kind of history dependence since the governing function is computed at a more recent time. A fractional delay differential equation (FDDE) combines two popular nonlocal mechanisms: the fractional memory integral and/or delayed arguments. The numerical treatment should include the estimation of both mechanisms and should include the history function when a delayed argument appears in front of the initial time [3,4].

Predictor-corrector schemes, finite-difference schemes, step schemes, collocation schemes, reproducing-kernel schemes, neural approximations, global or piecewise spectral expansions [4,5,8] are all included in the numerical literature. The operational-matrix methods are an important subclass of the spectral and wavelet methods. They solve the unknown solution, or its highest order derivative, numerically in a finite basis and approximate fractional integration, fractional differentiation, products and delay maps using matrix operations. Then Collocation or Galerkin testing reduces the FDDE to a system of linear or nonlinear algebraic equations. This reduction is convenient from a computational perspective, but truncation error, projection error, nonlinear-solver error, and poor conditioning of the basis (from the perspective of a poorly scaled basis) remain.

There are two typical formulations that have been given special consideration. Singh employed shifted 3rd kind Chebyshev polynomials to compute numerical solutions of FDDEs and provided some examples with error estimates, convergence observations and CPU times [13]. Genocchi wavelet functions and integer-derivative, fractional pseudo-operational, delay and pantograph matrices were created by Dehestani, Ordokhani and Razzaghi. The two methods have a common high-level goal, but have differences in their support, orthogonality, construction of operators and handling of delayed arguments.

There are three common mistakes to avoid making when making a scientific defensible comparison. As the first point, the convergence of some errors in a small number of truncation levels is not enough to conclude on the spectral convergence; one should assume analyticity or a similar regularity, otherwise the convergence should be algebraic [6,7]. Second, an exact delay matrix is exact for a finite dimensional representation and delay class for which it was constructed but not for arbitrary varying and/or state-dependent delays. Thirdly, the CPU times and error values of different studies may not be directly compared unless the equation, the fractional order, the history, the reference solution, the stopping criteria, the implementation and the hardware are controlled.

This review has four aims. It provides a correct mathematical definition of what is required for a coherent comparison; it presents the two methods in a common operator format; it offers a list of the strengths and weaknesses of both methods without identifying a universal winner of the methods which is not supported by the data; and it suggests a minimum standard for benchmarks which future comparisons can be made against. The contribution is a methodological one, and not empirical in the sense that new numerical values of performances are not claimed.

2. Scope and review method

This article is a critical narrative review and not a systematic review or meta-analysis. The selected terms fractional delay differential equation, fractional pantograph equation, operational matrix, spectral collocation, Chebyshev, and wavelet were used in various combinations to search in the publisher databases, DOIs, and indexed scholarly search results until 31 July 2026. The studies that considered constant [9] or proportional [25] or time-varying [9] delays were given priority; also, those that reported an approximation or convergence argument, or numerical comparisons with a clearly defined method, were given priority [9, 25].

Singh [13] and Dehestani et al. [11] were chosen as anchor studies to be compared since they are a global orthogonal polynomial construction and a compactly supported wavelet-like construction, respectively. Their claims were put in perspective using related work, such as Chebyshev and Bernoulli wavelet methods [9,10], shifted Gegenbauer and Legendre wavelet formulations [14,15], general operational-matrix methods [16,17], fractional-order Taylor and Chelyshkov bases [18,20,21], Chebyshev cardinal functions [22], Krawtchouk wavelets [23] and a wavelet Galerkin method [24]. To capture the latest trend of the field at the time of the review, a 2026 multi-term collocation study was added.

Both methods were evaluated by the following criteria:

(1) Mathematical description of the basis and the support;

2. Orthogonality/Non-orthogonality and associated conditioning issues;

3. Preciseness or anticipated nature of the fractional and delay operators;

4. Convergence assumptions behind convergence statements;

5. Data for history and arguments delayed before and after the computational interval;

6. Algebraic system size, sparsity and requirements for nonlinear solvers; and

7. Repeatability of reporting mistakes and cost of computation.

The errors tabulated and the time of execution of the papers reviewed were not pooled, since the implementation of each paper is not common and there is no common open protocol to be used as a benchmark. The conclusions are therefore based on published evidence, not on a synthetic numerical ranking, and on structure in mathematics.

3. Mathematical framework

3.1 Caputo FDDEs and history data

Let $m-1<\alpha<m$, where $m\in\mathbb{N}$. For a sufficiently smooth function x, the left-sided Caputo derivative is

$$D_{0+}^{\alpha,\mathrm{C}}x(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}\,x^{(m)}(s)\,ds.$$

If $\alpha=m$ is an integer, the Caputo derivative reduces to $x^{(m)}(t)$. For an integer power $q\geq 0$,

$$D_{0+}^{\alpha,\mathrm{C}} t^q = \begin{cases} 0, & q = 0,1,\dots,m-1, \\ \dfrac{\Gamma(q+1)}{\Gamma(q+1-\alpha)} t^{q-\alpha}, & q \geq m. \end{cases}$$

The exponent is q-α not q. The difference is significant for a fractional operator projected onto an integer order polynomial space [1,2].

A generalized form of a retarded Caputo FDDE is

$$D_{0+}^{\alpha,\mathrm{C}} x(t) = F\left(t, x(t), x\big(\eta_1(t)\big), D_{0+}^{\beta,\mathrm{C}} x\big(\eta_2(t)\big)\right), \qquad 0 < t \leq T,$$

Where $0 \leq \beta < \alpha$ and the delay maps satisfy $\eta_j(t) \leq t$. Constant delays use $\eta(t) = t - \tau$, whereas proportional or pantograph delays use $\eta(t) = qt$ with $0 < q < 1$. If $\eta_j(t) < 0$, the problem requires a history function

$$x(t) = \varphi(t), \qquad -\tau_{\max} \leq t \leq 0,$$

With the compatibility conditions specific to the order α. Assigning the delayed argument to be nonnegative alters the mathematical problem and thereby the initial delay interval is not included. Thus, any numerical method must clearly specify the method for putting in the history contribution [3,4].

3.2 A common operational-matrix representation

Let $\mathbf{\Phi}_N(t) = [\phi_0(t), \dots, \phi_{N-1}(t)]^T$ be a finite basis. The approximation is written as

$$x_N(t) = \mathbf{c}^T \mathbf{\Phi}_N(t).$$

An operational matrix is best interpreted through a projection Π_N onto the trial space. In general,

$$\Pi_N\big(D_{0+}^{\alpha,\mathrm{C}} \mathbf{\Phi}_N\big) = \mathbf{D}_N^{(\alpha)} \mathbf{\Phi}_N, \qquad \Pi_N(\mathbf{\Phi}_N \circ \eta) = \mathbf{H}_N[\eta] \mathbf{\Phi}_N.$$

The following are finite dimensional representations, not automatic identities. The first relation is only exact for integer-power polynomial bases, where the fractional power of each basis function is still in the trial space; this is not generally true for the integer-power polynomial bases since $t^{(q-\alpha)}$ is not a polynomial, when α is not an integer. Likewise, when the history interval is treated consistently, and composition with η leaves the finite trial space unchanged, then the composition is exact.

After substituting the approximation into the FDDE, a collocation method imposes the residual at nodes t_i,

$$R_N(\mathbf{c}; t_i) = 0, \qquad i = 1, \dots, N.$$

The linear FDDEs generate a linear algebraic system. For nonlinear FDDEs, an iterative solver is used and tolerances, initial guesses, and stopping rules are required and must be reported. Here is a summary of the common workflow as shown in figure 1.

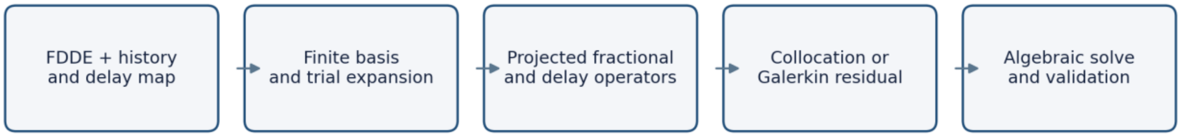

Figure 1. Common workflow of operational-matrix solvers for FDDEs. Every arrow can introduce approximation or conditioning error, so validation must be independent of the discretization used to construct the solution.

3.3 What constitutes a fair comparison?

Evaluating accuracy against an exact solution at the same fractional order is suitable. Or it is suitable to evaluate against an independent solution, for which it is proved to be more accurate than both solutions being compared, for a reference solution. The error of the fractional problem cannot be measured by comparing with just one of the integer order limiting solutions. A comparison should include, at a minimum, the errors in the L^2 and L^∞ norm of the solution, the error of the residual on a more densely discretized independent grid, the condition number (or a surrogate for it), and the setup and solve times of the individual components.

Convergence needs to be expressed conditionally also. Without breaking the rules of the Chebyshev approximation, a global Chebyshev approximation can be convergent in the geometric sense for an analytic solution. When the solution is weak, similar to a lot of Caputo problems, expanding the solution globally with integer powers of the argument could lead to an algebraic convergence of the series only after resolving the singularity via mapping, domain decomposition, singularity subtraction, or via a fractional-order basis [6,7,18,21]. No locality-based method can be guaranteed to be spectrally convergent or numerically stable, although a piecewise polynomial or wavelet representation can localize the loss of regularity.

4. Shifted third-kind Chebyshev operational-matrix method

4.1 Basis definition

The third-kind Chebyshev polynomials V_n on $[-1,1]$ may be defined by

$$V_n(\cos\theta) = \frac{\cos\left(\left(n+\frac{1}{2}\right)\theta\right)}{\cos(\theta/2)}.$$

They are orthogonal with respect to $[(1+\xi)/(1-\xi)]^{1/2}$. On $[0,T]$, the shifted basis is $\hat{V}_n(t) = V_n(2t/T-1)$, with weight proportional to $[t/(T-t)]^{1/2}$. On $[0,1]$ the correct recurrence is

$$\hat{V}_0(t) = 1, \qquad \hat{V}_1(t) = 4t-3,$$

$$\hat{V}_{n+1}(t) = (4t-2)\hat{V}_n(t) - \hat{V}_{n-1}(t).$$

This recurrence must not be confused with the fourth-kind Chebyshev family, whose first shifted polynomial, and weight are different. Confusion of the third- and fourth-kind definitions results in a different Gram matrix for the two cases, and destroys the orthogonality relation quoted in [7,19] for the derivation of coefficients.

4.2 Operator construction

Numerous operational-matrix solvers use the same integration formulation with the highest Caputo derivative being approximated first,

$$D_{0+}^{\alpha,C} x_N(t) \approx \mathbf{c}^T \widehat{\mathbf{V}}_N(t),$$

The solution is then reconstructed using the polynomial terms necessary from the initial conditions, and a Riemann-Liouville fractional integral. The fractional integral of each polynomial basis function can be expressed in monomials, and then projected onto the truncated Chebyshev space. This yields an N x N matrix with entries that will be a function of α and N [13].

To get a proportional delay, then the basis can be computed directly at the collocation nodes, at qt_i. If the delay is not constant, but nonlinear, as in the case of delayed evaluation, the known history can cross the initial point and the trial approximation must be replaced by this known history. Only delay maps that can be played with the selected interval and projection will be convenient to play from a single fixed delay matrix. The direct evaluation of nodes may be more intuitively understandable and does not imply accuracy if only interpolation occurs.

4.3 Strengths

There are a number of true benefits to the shifted third-kind Chebyshev approach. Orthogonality gives a principled projection and it generally has a better conditioning than a pure monomial basis. If the solution is smooth throughout the interval, then global polynomials are very efficient. The endpoint interpolation error can be controlled by Gauss or Gauss-Lobatto nodes which also are used for accurate quadrature. The recurrence allows to evaluate the basis of stable polynomials without explicitly computing the high-degree monomials. The principal operational matrices can be constructed just once and used for parameter studies, for a given fractional-order and interval.

4.4 Limitations and appropriate interpretation

The approach is a universal approach. All the coefficients were influenced by a local defect, a derivative discontinuity resulting from the delay history or an initial fractional-power singularity. The problem can then become more ill-conditioned before the desired level of asymptotic accuracy is achieved by increasing the degree of the polynomial. The error can also be hidden in the algebraic solve, as is the case for the operational matrix used for a noninteger derivative and integral.

An analysis of convergence does not guarantee the convergence of a particular coefficient decay to be exponential. The algebraic coefficient bound suggest algebraic convergence, but if stronger analyticity assumptions are made then the convergence is algebraic. The small numbers of smooth manufactured examples for which published error plots exist are evidence of performance on these examples, but do not provide any assurance of uniform superior performance of the basis for all FDDEs. Finally, delays that depend on time, on the state or on the history of the system require extra evaluation logic that cannot be expressed as a simple global translation matrix.

5. Genocchi wavelet method

5.1 Genocchi polynomials and localized functions

The generating polynomial can be used to introduce genocchi polynomials.

$$\frac{2ze^{tz}}{e^{z}+1}=\sum_{m=0}^{\infty} G_m\,(t)\frac{z^m}{m!},\quad |z|<\pi.$$

They form an orthogonal family under the standard L^2 inner product with polynomial sequence $G'_m(t)$ which satisfies $G'_m(t)=mG_{m-1}(t)$. The Genocchi wavelet construction focuses normalized versions of these polynomials on dyadic subintervals.

If $I_{k,n}=[(n-1)2^{-(k-1)},n2^{-(k-1)})$, a typical basis function has the form

$$\psi_{n,m}(t)=2^{(k-1)/2}\,\tilde{G}_m(2^{k-1}t-n+1)\mathbf{1}_{I_{k,n}}(t),$$

Where; $n=1,\dots,2^{k-1}$ and $m=1,\dots,M$. $N=2^{k-1}M$ [11] is the total number of functions. The functions defined on disjoint subintervals are orthogonal because the supports are disjoint. These different polynomial orders are not orthogonal, however, within the same subinterval. So it is more correct to say that the construction is a wavelet-like polynomial basis which is compactly supported than an orthonormal wavelet basis.

5.2 Operational and pseudo-operational matrices

Based on the polynomial differentiation relation and a transformation between localized functions and Genocchi polynomial coefficients, Dehestani et al. develop an integer-derivative matrix [11]. When the order of the Caputo derivative is not an integer the transformed functions will produce fractional powers not in the finite Genocchi space. Thus the resulting fractional derivative matrix is then explicitly a pseudo-operational matrix. The choice of terminology is significant because it is a good indicator of approximation, not algebraic closure.

The same structure is used for delay and pantograph operator matrices for certain constant and proportional transformations. These kinds of matrices may be algebraically exact with regard to the finite representation and also the assumed delay map. Their usage is more accurate when a shift traverses a dyadic interface, when the shift is at the history interval or if it is nonlinearly related to time or state. If so, the possibility of the support changing and the history term must be addressed directly.

5.3 Strengths

The Genocchi construction is a construction set with helpful structural aspects. Compact support means that the basis function evaluation is restricted to a subset of the coefficients, and the influence of reduced regularity is restricted. The resolution level k and polynomial order M are independent parameters which correspond to h- and p-refinement. Delay and pantograph transformations are included in the original formulation of classes considered in this study. For multi-domain treatment, naturally piecewise support is compatible, and if the operators maintain locality can support sparse assembly. Several fractional differential and partial differential equations can be solved by the same transformation machinery as the method extends to these.

5.4 Restrictions and correct interpretation

The local basis for the Genocchi functions is not orthogonal. High-degree nonorthogonal polynomial transformation matrices can be ill-conditioned and can cause roundoff to be amplified by any inverse transformation. This is a within-element problem that cannot be avoided with compact support. Besides, the Caputo derivative is nonlocal: given a local basis, the fractional derivative will in general have global memory. This means that the resulting fractional operator can then be dense, and thus not be as sparse as the suggestion from the support of the original basis.

The resolution parameters should also be reported transparently. It is important to note that a global polynomial degree N is compared with a Genocchi polynomial order M, but excluding the number $2^{(k-1)}$ of subintervals indicates an underestimate of the size of the wavelet system. For a fair comparison, the total degree of freedom, the density of the matrix and the nonlinear iterations should be considered. Last but not least, localization is useful only if the features of the solution are in fact localized. A fixed uniform dyadic partition is a multiresolution structure, but does not form an adaptive algorithm.

6. Critical comparison

Table 1 distinguishes two types of structural features: advantages and disadvantages. The table excludes unsupported conclusions like 'excellent', 'exact' or 'faster', depending on the problem or implementation.

Table 1. Structural comparison of the two anchor methods

Criterion	**Shifted third-kind Chebyshev method**	**Genocchi wavelet method**
Trial space	Global polynomials on the full interval	Piecewise localized Genocchi-polynomial functions on dyadic subintervals
Support	Global	Compact before application of the nonlocal fractional operator
Orthogonality	Orthogonal under the third-kind Chebyshev weight	Disjoint supports are orthogonal; Genocchi orders on one subinterval are nonorthogonal

Criterion	Shifted third-kind Chebyshev method	Genocchi wavelet method
Fractional operator	Dense projected integration/differentiation matrix	Fractional pseudo-operational matrix; generally dense because of memory
Delay treatment	Direct nodal evaluation or projected delay matrix	Explicit matrices for specified constant and proportional delays
History interval	Must be inserted when delayed arguments are negative	Must be inserted when a shift crosses the initial point
Expected advantage	Smooth, globally regular solutions and simple delay maps	Localized loss of regularity or known piecewise structure
Principal risk	Global sensitivity to singularities and high-degree conditioning	Nonorthogonal within-element basis and transformation-matrix conditioning
Degrees of freedom	Approximately the global polynomial degree plus one	$2^{k-1}M$; both level and local order must be counted
Refinement	Primarily global p-refinement unless a multi-domain extension is added	Uniform h/p control; adaptivity requires an additional estimator and refinement rule
Complexity claim	Dense direct solve is typically $O(N^3)$ and $O(N^2)$ memory	Same worst-case dense solve in total degrees of freedom; locality can reduce assembly only when preserved
Evidence-supported conclusion	Efficient for sufficiently smooth benchmark problems [13]	Accurate for the delay classes and examples studied in the source paper [11]

6.1 Accuracy and convergence

One of the methods has a smaller L^∞ error than the other at a given nominal order for no reason. The error is a combination of the error of best approximation, the error of operator projection, the error of delay evaluation, the error of collocation, the error of algebraic solution, and the error of finite precision. The global Chebyshev expansions may be very efficient for analytic functions [6,7]. A piecewise basis can provide a better distribution of degrees of freedom in the case of functions that are not regular except on some localized domain. However, the stability property of an orthogonal multiwavelet is not necessarily inherited to the Genocchi basis and the global Caputo operator can connect all subintervals.

To have a reliable convergence study, the total degrees of freedom should be changed, not just one basis parameter. It should include errors at the actual fractional order, a residual on an independent grid and should have sufficient resolution levels to differentiate algebraic and geometric decay. If the plot is on a semilog graph with three or four points, then a straight line is indicative but not conclusive evidence of spectral convergence.

6.2 Delay handling

It is usually more important to delay structure than basis family. Both methods have proportional delays which are easy to measure. If there is a piecewise rule before and after $t = \tau$, the solution will be determined using the same method as done before. One precomputed translation matrix may not exist if multiple, time-varying, distributed, neutral or state-dependent delays are present. In these situations, it is easier to understand the evaluation of the delay at the delayed collocation points (with accurate interpolation) and explicit history logic than to fit the delayed collocation points into a nominally exact matrix.

When the problem matches with the delay or pantograph matrix of the Genocchi construction, the construction is very interesting in the finite representation. When the delayed basis is computed directly at nodes, the Chebyshev method is as simple as well. Hence, a delay matrix cannot be regarded as a general tool for making accurate measurements.

6.3 Conditioning and computational cost

In the usual model, a dense direct algebraic solve has an $O(N^3)$ cost and an $O(N^2)$ storage. For nonlinear equations, this cost is multiplied by the number of iterations; and possibly requires assembly of the Jacobian. The assembly cost depends on the formula, quadrature, caching and reusing of the recurrence relations. The global Chebyshev matrices are generally sparse and are well known and stable to evaluate. The localized wavelet basis can be used to obtain sparse evaluation matrices, and sparsity can be lost due to the fractional memory operator and delay crossings.

Wall-clock time is subject to the precision, solver library, precision, hardware, warm-up, stopping tolerances, vectorization and programming language. In the absence of these conditions any table from one publication cannot be compared with a table from another publication. Such comparisons as "30–40% faster" are therefore unwarranted in the absence of a common implementation and controlled benchmark.

6.4 Evidence from related literature

Subsequent work has added diversity to the basis designs, rather than offering a clear winner (Table 2). The direction of the trajectory is towards using fractional-order bases for endpoint singularities, cardinal representations for easier evaluation and wavelet or multiwavelet bases for getting them sparse and localized.

Table 2. Critical appraisal of representative operational-matrix and related FDDE studies

Study	Basis or formulation	Main contribution	Limitation relevant to cross-study comparison
Saeed et al. [9]	Chebyshev wavelets with a step-based treatment	Combined delay handling with wavelet approximation	Benchmark and implementation choices differ from later matrix methods
Rahimkhani et al. [10]	Bernoulli wavelet operational matrix	Derived a fractional integration matrix for FDDEs	Nonorthogonal polynomial structure and study-specific tests complicate general ranking
Dehestani et al. [11]	Genocchi wavelets	Integer, fractional pseudo-, delay, and pantograph matrices with Sobolev error analysis	Nonorthogonality and total system size require closer conditioning and complexity reporting
Ali et al. [12]	Chebyshev spectral collocation	Operational matrix treatment of fractional-order delay equations	Evidence is based on the selected smooth examples and does not establish universal superiority
Singh [13]	Shifted third-kind Chebyshev polynomials	Error discussion, convergence observations, and numerical tests	Global basis; reproducibility is limited without shared code and a common benchmark
Yuttanan et al. [15]	Legendre wavelets with explicit fractional integrals	Avoided a separate projected integration matrix for the chosen wavelet functions	Comparisons still depend on problem-specific reference solutions and degrees of freedom
Avcı [18]	Fractional-order Taylor basis	Matched fractional-power behavior through a fractional basis	Basis and benchmark differ from integer-polynomial and wavelet studies

Study	Basis or formulation	Main contribution	Limitation relevant to cross-study comparison
Ahmed and Al-Sharif [21]	Fractional-order Chelyshkov functions	Targeted fractional-power solutions with fewer basis terms in reported tests	Published efficiency claims remain conditional on the selected examples and parameter choice
Jebreen and Dassios [22]	Chebyshev cardinal functions	Reduced operator evaluation through a Lobatto cardinal representation	Cardinal conditioning and performance need common independent benchmarking
Hadi et al. [24]	Wavelet Galerkin method after Volterra reformulation	Added convergence and stability analysis and threshold-based sparsification	Threshold accuracy-cost tradeoffs must be reported to compare fairly with dense solvers
Saw et al. [25]	Collocation for multi-term nonlinear FDDEs	Extended current work to multi-term nonlinear problems	Recent results reinforce method diversity rather than settle the global-versus-local comparison

7. Reproducibility framework for a valid numerical comparison

A head-to-head experiment should be prospective in a publishable experiment. Both solvers have to be based on the same mathematical problems, reference solutions, degrees of freedom, tolerances, software setting, and error definitions. A minimum reporting standard is given in Table 3.

Table 3. *Minimum benchmark and reporting protocol*

Component	Required specification	Rationale
Problem classes	Smooth manufactured solution; weak initial singularity; constant delay crossing $t = 0$; proportional delay; time-varying delay; neutral delayed derivative	Tests the structural claims rather than only the easiest smooth case
Fractional orders	At least one value in each relevant noninteger interval; integer limits reported separately	Prevents integer-order validation from being mistaken for fractional-order accuracy
History data	Explicit $\varphi(t)$, interval, compatibility conditions, and code path for $\eta(t) < 0$	Makes the delay problem mathematically complete
Degrees of freedom	Total coefficient count, subinterval count, local order, collocation nodes, and matrix density	Prevents unfair comparison of N with only the local order M
Reference solution	Exact solution at the same α, or an independently converged high-precision solution with documented tolerance	Avoids self-reference and integer-limit proxies
Error measures	L^∞, L^2, relative error where meaningful, and residual norm on an independent dense grid	Separates solution error from interpolation at collocation nodes
Conditioning	Matrix condition estimate, nonlinear iterations, solver tolerance, and precision	Explains stagnation and roundoff sensitivity
Timing	Separate assembly and solve time; median of repeated runs; hardware, language, libraries, and thread count	Makes cost measurements interpretable and reproducible
Availability	Source code, fixed random seeds if used, input files, and machine-readable outputs	Enables independent verification and reuse

The reference solutions cannot be generated by a similar method, only using a slightly larger truncation and simply treated as reference solutions without a convergence certificate. If there is no closed form, then one should use an independent method, use more precision, or use interval bounds, or conduct a Richardson type study with a significantly increased number of resolutions. Once the collocation points have been determined, it is important to evaluate the residuals away from the collocation grid since the collocation procedure can cause the residual to be artificially small at points where it is zeroed out.

8. Method-selection guidance

Deciding which of the two approaches (global Chebyshev or localized Genocchi) to use should be done after, and not prior, to inspecting the problem. Simple proportional delay with smooth solution and short interval. The shifted Chebyshev method is typically the first option that comes to mind being the more economical approach: it is global polynomial convergent and the basis is orthogonal. Known local irregularity and/or interface. If the partition can line up with the feature, it is better to use piecewise or wavelet method. The advantage should be validated with the total number of degrees of freedom and conditioning.

Fractal singularity of the origin of fractional power. By itself, a high degree global integer-polynomial basis and an unmodified local Genocchi basis are not necessarily optimal. A fractional-order basis, singularity subtraction, graded mapping or a spectral-element treatment should be considered [18,21].

Frequently delaying crossing the first point. Use a piecewise formulation where the history is simply added in between $0<t<\tau$; do not extrapolate the calculated solution into the negative time.

Delayed response - based on time or state. Evaluate directly instead of via a translation matrix, and verify interpolation, rather than relying on pre-assumptions of exactness in the translation matrix.

Large systems and/or repeated parameter studies. When assembling the fractional operator use matrix elements and compare sparsity after the operator is assembled, cache order-dependent matrices, and use iterative or structured solvers after proving stability.

This guidance is purposely 'rule of thumb'. It does not provide a sure winning position; it is a defensible starting position.

9. Research gaps and future directions

9.1 Regularity-aware adaptivity

In most operational-matrix studies the order of truncation and/or the level of the wavelet are chosen in advance. A better method would be to estimate the local residual or coefficient decay, refine only where necessary and separate out these two memory properties, which are spatial localization and the global memory coupling of the Caputo derivative. A multi-domain Chebyshev polynomial and a stable orthogonal multiwavelet basis strategy is more promising than just concatenating two existing matrices for an hp strategy.

9.2 Conditioning-aware analysis

The coefficient bounds are not sufficient to ensure a reliable calculated coefficient. Future research should include reports of Gram-matrix spectra, estimates of the operators' condition, error in the back solution and sensitivity of delayed evaluation. For nonorthogonal localized polynomial families, two possibilities are to consider: the formulation via the original transformation or the possibility of achieving stable orthogonalization or well-conditioned local bases.

9.3 General delay operators and history interfaces

Proportional and constant delays are merely part of the story of FDDEs. For delays that vary with time, depend on the state, are distributed and/or neutral, rigorous operator constructions are required and explicit handling of each interface point where the delayed argument is passed to the history or mesh. The approximation error due to the delay operator should be distinguished from the fractional-operator error.

9.4 Reproducible benchmark library

There is no common benchmark suite of precisely defined fractional-order manufactured solutions, non-smooth cases, history standardization and reference implementation. A public library should symbolically store the equations; have an automatic generation of forcing terms; evaluate errors on fixed independent grids; and record the density and conditioning of matrices together with runtime of the computations. With such a resource it would be possible to make methodological statements that could be disproved.

9.5 Fast and structure-preserving solvers.

Some recent wavelet Galerkin related research on thresholding to reduce the algebraic cost is considered in [24] Future work should investigate the error associated with the thresholding and attempt to quantify this, as well as consider hierarchical matrices, sum-of-exponentials approximations to the Caputo kernel, block preconditioners and matrix-free delayed evaluation. The speed should be measured at a fixed accuracy and not at a fixed polynomial order (nominal).

10. Conclusion

Shifted third kind Chebyshev polynomials and Genocchi wavelets are helpful and structurally different tools to deal with Caputo FDDEs. Chebyshev method shows an orthogonal international representation, and is forcing for straight solutions on average intervals. The Genocchi method gives a polynomial basis of the trial functions supported by a set of compact elements, and provides explicit constructions of the matrices for several types of delays; however, the polynomial basis is nonorthogonal and the Caputo operator is approximated. Both sets of properties do not imply generality of superiority.

The main point that is made in this review is methodological since the published errors and CPU times are significant only within the context of the original problem and application if a controlled common benchmark is not provided. The total degrees of freedom should be counted, the history interval should be treated explicitly, the distinction between exact and projected operators should be made clear, the regularity assumptions should be made clear, the independent residuals should be evaluated and information on conditioning and timing should be disclosed. At those levels, basis selection is a clear tradeoff on the regularity of the solutions, and on the structure of delays, not based on tables that are incomparable.

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