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On N-Feebly Open Set in Topological Spaces

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Abstract

In this research, the concept of N-feeble-open (N-feeble-O) sets in topological spaces is introduced and studied and the concept of N-feeble-O sets is used to construct a new class of compact spaces. Several fundamental properties of N-feeble-O sets are proved, such as relationships with feeble-open sets, semi-open sets and N-semi-open sets. Topological concepts that are related to this one, including interior, closure, derived sets, adherent points and subspaces, are also explored. A new form of N-feeble compactness is defined, based on the newly introduced N-feeble-O sets, and certain fundamental properties of this compactness are explored. Relationships between the N-feeble compact spaces and other compactness notions, namely, compactness, semi-compactness and feeble compactness, are investigated. A number of theorems, propositions, corollaries and illustrative examples are given to get the results obtained and to clarify the differences between these classes. In addition, preservation properties with subspaces, closed subsets and intersections are explored. The results add to the generalized topological concepts and lay foundations for future research of N-feeble-O sets, compactness and associated topological functions.

Keywords

Feeble-O set (briefly, f-O), feeble-c set (for a short time, f-closed), N-feeble open (briefly, N_f-O), Set, N-feeble closed (simply, N_f-closed) and semi-compact top. sp.s (simply, s-compact top sp).

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1. INTRODUCTION

In the top. sp. the semi open set defined and examined some properties of s-continuous functions in 1963, by Levine [1]. In 2014 J. M. J. [2] defined novel class of semi-O sets, explicitly S_S-open set and gives several properties about this set. In 2020 A. A. Salih [3] introduced new idea of sets baptized N-semi-O set (briefly Ns-O set) in a top. sp.s. In 1985 Jankovic, D. S. [4] introduced new idea of sets baptized feeble-O set (briefly f-O set) in a top. sp.s. IN 2023, Harith M. Abdul Razzaq, and Haider J. Ali [5], introduced new idea of sets baptized m_x-N-semi-O set (briefly m_x-Ns-O set) in a minimal structure space. In 2025 Malik, Abdulkarim Hatem, and Haider Jebur Ali [6], introduced the concepts of f_b -boundary, f_b -interior, and f_b -closure. Their study specifically examined the f_b -boundary behavior for unfilled sets within the context of f_b -O set. In this work we intedused new idea of sets baptized N-feeble-O set (briefly N_f-O set) and studied all properties related by N_f-O sets and using this notion to introduce new class of semi-compact functions which introduced in 2000 by D. A. Ressen [7], namely N_f-compact functions. Also, we studied certain types of the operation on this function as restriction and composition.

2. PRELIMINARIES

Definition (2.1)[14]: For a space (X, τ) the subset G known as the S-O set if it is $G \subseteq Cl(\mathrm{Int}(G))$. Complementary S-O sets are known as s-csets. The s-c subset G of X indicated via $\mathrm{sCl}(G)$ isthe intersection of wholly s-c sets including G..

Definition (2.2) [13]: A subset Q ⊆X is classified as N_s-O if, for each x belong to Q, there exists an S-O set U covering x, s.t, the cardinality of U \Q is finite. Furthermore, a set is defined as Ns-c if and only if its accompaniment in X is an Ns-O subset.

Definition (2.3)[4]: We classify $A \subseteq X$ AS an f-O set if there exist some $G \in \tau$ fulfilling the INCLUSION $G \subseteq A \subseteq \mathrm{sCl}(G)$). The class of f-c sets of those subsets ehose complement s are f-O.

Definition (2.4)[1]: We can said aset A in sp X is ∝-O if $A \subseteq (CL(A^{\circ}))^{\circ}$

Definition (2.5)[8]: If theclosure of wholly open sets in X is open, aspace (X, τ) is well-known as extremely disconnected sp.

Theorm (2.6)[9]: Spouse τ be any topology on aset X. Family of each s-O sets of a space X constitutes topology if f (X, τ) extremely disconnected sp.

Definition (2.8) [11]: A top sp (X, τ) know as S-compact if all S-O cover of X admit to finite cover.

Proposition (2.7)[10]:

1. Supposing that $(\mathbf{Y}, \boldsymbol{\tau}')$ is s-open sub space of top sp $(\mathbf{X}, \boldsymbol{\tau})$. If A is s-comp. in $\mathbf{X}$, $\mathbf{A} \subseteq \mathbf{Y}$, thus A is S-comp. set in Y.
2. Consider that $(\mathbf{Y}, \boldsymbol{\tau}')$ is subspace of top sp $(\mathbf{X}, \boldsymbol{\tau})$. Ais S-compact set in$\mathbf{X}$. If A is S-compact set in$\mathbf{Y}$, **A isa subset of Y**.

Definition (2.9) [12]:

The space (X, τ) is referred to as feeble compact when each ever f-O covering of X possesses a finite sub-family that remains an open cover.

3. RESULTS AND DISCUSSION

Definition (3.1) :If there is an f-O set U that contains x, then for each $x \in Q \subseteq X$, therefore Q is known as N_f-O. Resulting a set $U \setminus Q$ is finite. It is raised to as an $N_f - \mathrm{c}$ set if it is the complement of N_f-O subset.

Remark (3.2):

1. The class of open sets is firmly contained within both the feeble open and N_f-O collections
2. Any set that satisfies the condition of being feeble open is necessarily a semi-O set
3. Every member of the N_f-O family is also an element of the Ns-O class.

Example (3.3):

1. Cogitate the set X={1, 2, 3, 4} equipped with the toplogy $\tau = \{X, \emptyset, \{1\}, \{2\}, \{1,2\}\}$. One can observe that while the subset Q={1, 3} qualifies as both N_f −O and feeble open, it fails to be an open set in (X, τ).
2. $\tau = \{X, \emptyset, \{a\}\}$ Consider a finite set X={a, b, c} equipped with the topology, with we define a subset A={a, b} which serves as an illustration of a semi-O set that fails to meet the criteria be feeble open set.
3. Consider the X be the set of positive integers Z^+ endowed with the topology $\tau = \{X, \emptyset, \{1\}\}$. The subset Q=X-{2} is found to be Ns-O; nevertheless, it does not belong to the class of N-feeble open set.

Proposition (3.4): Let (X, τ) be a top. sp. and let $\{U_a, a \in \wedge\}$ be an arbitrary assemblage of N_f −O subsets in X. Then, we demonstrate the validity of the following claims are valid:

1. The union $\bigcup_{\alpha\in\Delta} U_\alpha$ maintains the property of being an N_f −open set
2. The intersection $\bigcap_{\alpha\in\Delta} U_\alpha$ constitutes an N_f −closed set.

Proof:

1. Taking x pelong to $\bigcup\{U_a, a \in \wedge\}$ implies that x belongs to U_{ai} for at least one alphabetical listing $a_i \in \wedge$. Given the N_f −o of U_{ai}, we can find a feeble open set V such that $x \in V$ and $V \setminus U_{ai}$ is a limited set. Meanwhile $U_{ai} \subseteq \bigcup\{U_a, a \in \wedge\}$, at that time $(\bigcup\{U_a, a \in \wedge\})^C \subseteq (U_{ai})^C$. Consequently the intersect V with $(\bigcup\{U_a, a \in \wedge\})^C \subseteq V \cap (U_{ai})^C$. Henceforth $V \setminus \bigcup\{U_a, a \in \wedge\} \subseteq V / U_{ai}$. From the time when $V \setminus U_{ai}$ Is finitely populated, therefore $V \setminus \bigcup\{U_a, a \in \wedge\}$ is also finite. Henceforth $\bigcup\{U_a, a \in \wedge\}$ is N_f −o set.
2. Clear by (1).

Remark (3.5): To illustrate that The binary intersection of sets N_f −open sets might fail to be N_f −open.

Example (3.6): Ponder the top. sp. (X, τ) where X = {a, b, c, d} and the topology is defined as τ= {X, { }, {a}, {b}, {a, b}}. Let us take two subsets U = {a, c} also V = {b, c}. It can be verified that both U in addition V is N_f −o sets. However, their meeting U∩V equl{c} is not an N_f −o set in (X, τ), which approves that the intersection of N_f −o sets are not essentially preserve

Lemma (3.7): The f-open property is preserved when a set is intersected with an open member of the topology.

Proof: Consider an f-o set V subset of X alongside an open member O belong to (X, τ) within the given topological framework, to show that W=V∩O f-O set, we recall that the f-O of V implies the existence of an open set U satisfying $U \subseteq V \subseteq sCL(U)$. By intersecting these inclusions with the open set O, we obtain $U \cap O \subseteq A \cap O \subseteq sCL(U) \cap O$ Taking into account the topological property that $sCL(U) \cap O \subseteq sCL(U \cap O)$.

for any open set O, it follows that $U \cap O \subseteq A \cap O \subseteq sCL(U) \cap O$ Since U∩O is open, W meets the criteria for being an f-O set.

Then we must $U \cap O \subseteq V \cap O \subseteq sCL(U) \cap O \subseteq sCL(U \cap O)$, so $U \cap O \subseteq V \cap O \subseteq sCL(U \cap O$, now from the time when $U \cap O$ are open sets thus $U \cap O$ open set, therefore W= $V \cap O$ feeble-o set.

Proposition (3.8): The property of being N_f −open is preserved under intersection with any open set.

Proof: Assume Q isan N_f −o set and O is an open set in aspaceX, to prove O∩ Q. Take an arbitrary element x from O∩Q. Since Q is N_f −open, thereexists a f-O set V containing x such that V \ Q is finite. Since O is open, and V is f-O set. By lemma (3.7), $V \cap O$ is a f-O set containing x and ($V \cap O$) \ (Q∩ O) = (V\Q)∩ $O \subseteq V\backslash Q$, now since $V\backslash Q$ is finite thus ($V \cap O$) \ (Q∩ O) is finite, from this ($O \cap Q$) is N_f −open set.

Proposition (3.9): If the family of wholly feeble-O sets of aspaceX constitutes a top. On X, therefore the family of N_f −open establishes a top. On X.

Proof: It is evident that both empty set and X belong to $N_f O$(X) Furthermore, according to **Proposition (3.4)** the collection $N_f O$(X) is closed under arbitrary unions. To comprehensive the verification, it suffices to demonstrate that the intersection of any two N_f −o sets remains N_f −o. suppose A and B are two N_f −o sets and x∈ A⋂B, then x ∈ A and x ∈ B, Subsequently, there are f −o sets U_a and U_b are containing x, suchthat $U_a \backslash A$ in addition to $U_b \backslash B$ are finite sets. Since FO (X) is a topology onX, we must $U_a \bigcap U_b$ is f-O set. Now ($U_a \bigcap U_b$) \ A⋂B = ($U_a \bigcap U_b$) ⋂ ($A^C \bigcup B^C$) = ($U_a \bigcap U_b \bigcap A^C$) ⋃ ($U_a \bigcap U_b \bigcap B^C$) ⊆ ($U_a \bigcap A^C$) ⋃ ($U_b \bigcap B^C$). Accordingly ($U_a \bigcap U_b$) \ A⋂ B is a finite set. Henceforward A⋂B isan N_f −o set.

Corollary (3.10): The family of wholly N_f −open set inX establishes a topology on X. If (X, τ) is extremely disconnected space.

Proof: By previous theorem (3.9).

Proposition (3.11): If a space X is a discrete topological space, then FO(X) = N_fO(X).

Proof: Clear.

Remark (3.12): It should be noted that N_f −closedness is not preserved under the formation of unions of two such set

Example (3.13): Consider the standard top. sp. (R, τ_u). Let A=(1, 2] and B=[2, 3) be two subsets of R. These sets are private as N_f −open, because any half-O interval in the usual topology is feeble open, and subsequently N_f −open set. Their respective complements, R \ (1, 2] and R \ [2, 3) are therefore N_f −closed sets. However, their union: R \ (1, 2] ∪ R \ [2, 3) = $(-\infty, 2) \cup (2, \infty)$ =R \ {2} fails to be N_f −closed, this is due to the fact that the singleton {2} is not an N_f −open set, as every f-O set containing the point 2 is automatically infinite.

Lemma (3.14): impose (Y, t_Y) be a subspace of a space (X, t), and A ⊆ Y. Then

1. If A belog to FO(X, τ_X) thus A∈ FO(Y, τ_Y)

2. If A belog to FO(Y, τ_Y) where Y∈ FO(X, τ_X), therefore A∈ FO(X, τ_X)

Proposition (3.15): force (Y, τ') is feeble open subspace of a space (X, τ). If A ∈ N_fO(Y) A ⊆ Y, at that juncture A ∈ N_fO (X).

Proof: Assume A is a N_f −o set inY. For any x ∈ A, the definition of N_f −openness in Y Implying that Y contains a feeble open set U as a subset such that the set difference U \ A is finite set. Given that U is feeble-o in Y and Y itself is a feeble open subset of X, it follows from Lemma $(3.14)_2$, that U is also feeble open in the broader space X. Accordingly, A satisfies the criteria to be an N_f −open set in X.

Proposition (3.16): Let (Y, τ') be subspace of top. sp. (X, τ). If A ∈ N_fO (X) A ⊆ Y, then A ∈ N_fO (Y).

Proof: Suppose that A is a N_f −open subset of X. For each x ∈ A, the N_f −openness of A in X guarantees the existence of an f-O set U in X such that U \ A is a finite set. Prearranged that U is f-O in X, applying Lemma $(3.14)_1$, implies that U is also feeble open in the subspace Y. It follows, therefore, that A qualifies as an N_f −open set within Y.

Definition (3.17): "For any subset U in a topological space X, we define its N_f −interior, symbolized by N_f −int(U), as the set of all points x b; ong to for which an N_f −op neighborhood V can be found satisfying $x \in V \subseteq U$.

Example (3.18): In discrete top. sp., the points of any subset are an N_f-interior points.

Definition (3.19): Presume that A isa set ina sp X. Any arbitrary element as x belong to X is assumed tobe N_f −limit point of A, if $U \cap (A \setminus \{x\}) \neq \emptyset$, For all N-feeble-o set U contains x. N_f −derived set of A is the set of each N_f −limit points of A, and indicated by $N_f - d(A)$.

Exampl (3.20): Let (N, τ_{cof}) denote the co-finite top. sp. defined on the set of natural numbers. It follows that any point x in N , excluding the point {2} serves as an N_f-limit point of any subset $G = N - \{2\}$ of X, since each N_f-O containing x is infinite set. So $N_f - \text{limit}\,(G) = N - \{2\}$.

Definition (3.21): A point x that is a member of X is described to as a N_f −adherent point of G, G ⊆ X. If each N_f −o set V comprise x traversed with G. (i.e. V∩G ≠ Ø), the traditional of all N_f −adherent points for A is signposted by N_f-adh(G) or N_f-cl(G).

Example (3.22): Let X = {4, 5, 6} and τ = I. Obviously 4, 5 ∈ X are N_f −adherent points for the set A={4,5}.

Definition (3.23): **Definition (3.23)**: Presume That Q is a set in space X, now:

1. The N_f −interior of Q is demarcated as the collection of all N_f −o sub sets delimited Within Q.
2. N_f −closure (Q) = ∩ {K: Q ⊆ K, K is N_f −closed}.

Proposition (3.24): A subset Q of a top. sp. (X, τ) is N_f −o if and only if every one-point x belongs to Q, prevailing N_f −o set B_x such that $x \in B_x \subset Q$.

Prof: Suppose Q is an N_f −o set and let x ∈ Q. By defining B_x = Q, it follows that B_x is an N_f −open neighborhood of x such that the inclusion $x \in B_x \subset Q$ hold naturally.

Conversely. Based on the given hypothesis, the set Q can be exemplified as the union of the sets B_X, for all x ∈ Q, such that $Q = \cup_{x \in Q} B_x$. According to Proposition $(3.4)_1$, the union of any collection of N_f −o sets are itself N_f −o. Therefore, it monitors that Q is an N_f −o set

Proposition (3.25): impose U be a set in a space X, therefore:

1. U is an N_f −open set iff N_f −interior (U) = U.
2. U is an Ns-closd set if f N_f −cl(U) = U.

Proof:

1. Since the union of all N_f −o set is the N_f −o set, then $\mathrm{N}_f - \mathrm{int(U)}$ isthe largest N_f −op set containedin U. In the meantime U is N_f −o set, at that time $\mathrm{N}_f - \mathrm{int(U)} = \mathrm{U}$. In opposition, on every occasion $\mathrm{N}_f - \mathrm{int(U)} = \mathrm{U}$, at that juncture U is N_f −o set, (from the time when $\mathrm{N}_f - \mathrm{int(U)}$ is an N_f −op set).
2. Owing to the fact that the intersection of any family of N_f −c set is N_f −c set, then N_f −cl(U) is the minimum N_f −c set encompasses U. Meanwhile U is an N_f −closed set, at that moment N_f −cl(U) = U. In opposition, at whatever time N_f −cl(U) = U, at that juncture U is an N_f −c set, (in the meantime N_f −cl(U) is an N_f −c set).

Proposition (3.26): Consider U, V are a subset of a space X and U⊆V, at that moment:

1. The N_f −interior of a set U is always contained within the set itself
2. The N_f −interior of any subset U constitutes an N_f −open subset within the space X.
3. $Int_{\mathrm{N}_f}\left(Int_{\mathrm{N}_f}U\right)\, EQUAL\; Int_{\mathrm{N}_f}(U)$
4. Int_{N_f}(X) $EQUAL$ X and Int_{N_f}(Ø) $EQUAL$ Ø.
5. Int_{N_f}(U) sub set Int_{N_f}(V).

Proof: Clear.

Propostion (3.27): Conceder U, V are sets in a space X and $\mathrm{U} \subseteq \mathrm{V}$thereafter:

1. $\mathrm{U} \subseteq \mathrm{N}_f - \mathrm{cl(U)}$.
2. $\mathrm{N}_f - \mathrm{cl(U)}$ isan $\mathrm{N}_f - \mathrm{closed}$.
3. $\mathrm{N}_f - \mathrm{cl}(\mathrm{N}_f - \mathrm{cl(U)}) = \mathrm{N}_f - \mathrm{cl(U)}$.
4. $\mathrm{N}_f - \mathrm{cl}(\emptyset) = \emptyset$ and $\mathrm{N}_f - \mathrm{cl(X)} = \mathrm{X}$.
5. IfU $\subseteq$ V, thus $\mathrm{N}_f - \mathrm{cl(U)} \subseteq \mathrm{N}_f - \mathrm{cl(V)}$.

Prof: Clear.

Proposition (3.28): A set M in a top space X is N-feeble-c iff $(\mathrm{N}_f - \mathrm{limit(M)} \subset \mathrm{M})$.

Proof: Assume M is an N_f −c sub set of X and let x∈ X \ M, This implies that x belongs to the complement M^C which is N_f −open by definition. Consequently, there occurs an N_f −o neighborhood B_x such that x $\in B_x \subset M^C$. It follows that $B_X \cap M = \emptyset$ which demonstrates that x cannot be an N_f −limit point of M. Since no point outside M is an N_f −limit point, we conclude that the set of all N_f −limt points is contained within M, such that N_f −limit(M) ⊂ M.

Proposition (3.29): Let A bea subset ofa top. sp.X, then N_f −cl(A) = A ⋃ N_f −d(A).

Proof: Since both the N_f −derived set of A's, and and the set A itself are contained within the N_f −closer (A). it follows that their union satisfies the inclusion N_f −d(A) sub set N_f −cl(A) To establish the reverse inclusion, N_f −cl(A) sub set A ⋃ N_f −d(A). In the meantime, N_f −cl(A) is the smallest N_f −c set containing A. Then all that vestiges is for us to do is prove that $A \bigcup \mathrm{N}_f - d(A)$ is N_f −c set. assume $x \in X \setminus (A \bigcup \mathrm{N}_f - d(A))$, formerly $x \notin A \bigcup \mathrm{N}_f - d(A)$ and thus $x \notin A$ and $x \notin \mathrm{N}_f - d(A)$, consequently, there happens an N_f −o neighborhood Ux of x that has no elements of A besides x itself. However, since x is nt belong to A, it follows that U_X is entirely disjoint from A. This implies that $U_X \subseteq$ X \ A. Furthermore, no element indoors the neighborhood U_X can be N_f −limit point of A. Thereforeno point of U_X can belong to N_f −d(A). Thus $x \in U_X \subseteq X \setminus N_f$ −d(A). Henceforth $x \in U_X \subseteq X \setminus A \cap X \setminus N_f$ −d(A) = X \ (A $\cup N_f$ −d(A)), Consequently, $x \in U_X \subseteq X \setminus$ (A $\cup N_f$ −d(A)). Therefore X \ (A $\cup N_f$ −d(A)) is N_f −o. Hence A $\cup N_f$ −d(A) is N_f −c. For that reason N_f −cl(A) $\subseteq$ A $\cup N_f$ −d(A).

Propostion (3.30): Assume X isa space and A isa set inX. A point $x \in X$ belongs to the N_f −cl of A if and only if each N_f −o environment U_x centered at x possesses at least one point in common with A. This is equivalent to proverb that $U_x \cap A \neq \emptyset$ for every one N_f −o set U_X containing x.

Proof: Assume that $x \notin N_f - cl(A)$. By definition, the N_f −closure of A is the intersection of all N_f −c supersets of A: N_f −cl(A)=$\bigcap${B: B is N_f −c set in X, $A \subseteq B$}. If x is not in this intersection, there must exist at least one specific N_f −c set B such that A subset of B and x is not belong to B Consequently, x belongs to the complement X \ B, which is an specific N_f −open set. Let U_x = X \ B Since A ⊆B, it shadows that: $U_x \cap A$ = (X \ B) ∩A⊆ (X \ B) ∩B = ∅ Thus, we have found an N_f −o set U_x encompassing x such that $U_x \cap A$ = ∅, which completes the necessity part of the proof. On the contrary, suppose that existing an N_f −o set U_X containing x in which $U_X \cap A = \emptyset$. So $A \subseteq (X \setminus U_X)$ and in the meantime x not belong to therefore X \ U_X, in consequence $x \notin N_f$ −cl(A).

Propostion (3.31): **Assume U isa set in a spaceX., then:**

1. $X \setminus N_f - \text{closure}(u) = n_f - \text{interior}(X \setminus U)$.
2. $X \setminus N_f - \text{interio}(U) = N_f - \text{closure}(X \setminus U)$.
3. $N_f - \text{closure}(U) = X \setminus N_f - \text{interior}(X \setminus U)$.
4. $N_f - \text{interior}(U) = X \setminus N_f - \text{closure}(X \setminus U)$.

1. N_f −cl(U) = $\bigcap${B: B is N_f −c set inX, U$\subseteq$B}, by definition: Taking the complement $X \setminus N_f -$ closure (U) equle to X \ [$\bigcap${B: B is N_f −c set inX, U$\subseteq$B}] = ∪{X \ B: X \ B isN_f −o set in X, $X \setminus B \subseteq X \setminus A$ }= N_f −int (X \ U). The same contextual reasoning is applicable across all cases

Propostion (3.32): A set A within the sp. X is considered N_f −open precisely when every element x ∈A is associated with a feeble-O environment U_x that, excluding a finite number of points M, is entirely contained in A.

Proof: Let A be an N_f −o set. For any element x belong to A there is a feeble-O set U_X containing x for which the complement of A relative to U_x is finite. By defining the finite set M as $U_x \cap (X \setminus A)$, we directly obtain the inclusion $U_x \setminus M \subseteq A$. This confirms the existence of a finite set M and a feeble-O environment U_x as specified. Conversely, assume $x \in A$. Consequently, according to our assumption, one can find a feeble open set U_X having x and a finite set M such that $U_X \setminus M \subseteq A$. Consequently $U_X \cap A$ is a subset of M Given the finiteness of M, the set $U_X \cap A$ must also be finite which is the requisite illness for A to be N_f −o

Proposition (3.33): Suppose X is a top. sp. and M sub set of X is N_f −c. It shadows from the duality of N_f −open sets that M can be pigeonholed by a feeble-c set B and a finite set K. Specifically, M can be represented as the intersection of B and the complement of some finite set K.

Proof: Tenancy agreement M is an N_f −closed, at that time X \ M isan N_f −op and therefore by above prop. for each $x \in$ X \ M, there exists feeble-op set U comprehending x and a finite set K such that $U \cap K^C \subseteq M^C$. It follows that M is contained within the complement of (U \ K) = X \ (U ∩ (X \ K)) = (X \ U) ⋃ K. If we define B as the complement of U, such that B = X \ U. then B is a feeble-c set. Consequently, the set M can be represented as a subset of the union B ⋃ K.

A new refinement of compactness, termed N_f −**compactness**, is developed here through the application of N_f −**o families**. This classification serves to provide a more robust structural requirement compared to f-compact spaces

Definition (3.34): Compactness in the N_f −sense implies that any N_f −o covering of the space X can be refined to a finite sub-covering.

Proposition (3.35):

1. The property of N_f −compactness is a stronger condition that implies standard compact, f −compact and N_s −compact space.
2. Every space that satisfies the N_s −compact criteria is necessarily compact, s-compact and f −compact.

Proof: Let $\{\mathrm{U_a}: a \in \wedge\}$ bean op cover of (X, τ). In consequence $\{\mathrm{U_a}: a \in \wedge\}$ be an N_f −open insurance of X. Now subsequently (X, τ) is an N_f −compact, at that juncture $\{U_a: a \in \wedge\}$ abstains a finite sub cover $\{\mathrm{U_a}$: a = 1, 2,... n} of (X, τ). At that moment the resilient is over. By the same context prove the second case.

Remark (3.36): However, the converse fails to be true in a general setting, as illustrated by the following example.

Example (3.37): We define a topology τ on the real line ℝ such that a subset U is considered open if it is either the entire space itself or if it avoids all elements of the natural numbers ℕ $\tau = \{U \subseteq P(R): U \cap N = \emptyset\} \cup \{\mathbb{R}\}$. Therefore (R, τ) is comp sp, s-compact and f-compact sp., but it's not N_f −compact and N_s −compact sp.

Proposition (3.38): Supposing that Y is a feeble-O subspace of a top. Sp. X. For any set A such that A⊆Y⊆X, the property of being N_f −compact is preserved when moving from the global space X to the subspace Y

Proof: Impose A is N_f −comp. set inX, to illustration A is N_f −compact set inY. impose {U$_i$: i∈I} is N_f −open concealment of A inY, thus {U$_i$: i∈I} isan N_f −open cover ofA inX (by Proposition (3.14)$_2$ Since A is N_f −comp. set inX, thus there is I'⊆I, so $A \subseteq \cup_{\mathrm{i} \in \mathrm{I}'} U_i$.Therefore A is N_f −compact set in Y.

Proposition (3.39): Tenancy agreement (Y, τ_Y) be a subspace of a top. sp. (X, τ_X). If a subset A⊆Y is N_f −comp. set relative to the subspace Y. then it is necessarily N_f −compact relative to the global space X.

Prof: assume G∈ N_f −Comp(Y), to verify G ∈ N_f −Comp(X). Tenancy agreement {U$_i$: i∈I} is N_f −open hiding place of G in X, thus {U$_i$: i∈I} is N_f −open cover of G inY (by Proposition (3.14)$_1$. Given that G is N_f −comp. virtual to Y, there exists a finite subindex set I' ⊆I such that $G \subseteq \cup_{\mathrm{i} \in \mathrm{I}'} \mathrm{U_i}$ This finite sub collection effectively covers G in X, thereby establishing that G is an N_f −compact set in the global space X.

Proposition (3.40): In an N_f −compact environment, any subset that is N_f −closed inherits the N_f −compactness property.

Prof: Let (X, τ) be an N_f −comp.t sp. in addition A is an N_f −c subset of (X, τ). Let $\{\mathrm{U_a}: a \in \wedge\}$ beany cover of A byN_f −open sets of X. Since A is N_f −closed, its complements X\A is an N_f −open set in X. From this the collection $\{\mathrm{U_a}$: $a \in \wedge\}$ ⋃ {X\A} is an N_f −open cover of entire sp. At this time since X isan N_f −compact, there exists a finite sub set $\wedge_1$ of $\wedge$ such that X= $\bigcup\{U_a: a\in \wedge_1 \bigcup X \setminus A\}$. By way of A ⊆X and X\A concealments no fragment of A, then $\{U_a: a\in \wedge_1\}$ isa finite sub cover of A. For that reason A is an N_f −comp. set.

Corollary (3.41): All closed subset of anN_f −compact space is an N_f −compact.

Corollary (3.42): Every f-c subset of an N_f −compact space is an N_f −compact.

Remark (3.43): The proofs of Corollary (3.41) and Corollary (3.42) follow directly from the fact that every closed, feeble closed are N_f −closed set.

Proposition (3.44): Let (X, τ) be a top. sp. If A is N_f −comp relative toX and B is anN_f −c subclass of X, at that point the intersection of A with B is N_f −comp. relative to X.

Prof: Impose A is an N_f −comp. relative to X. Lease $\{U_a: a \in \wedge\}$ is slightly shelter of (A ∩ B) by N_f −open sets of (X, τ). Therefore $\{U_a: a \in \wedge\} \bigcup B^C$ is a N_f −op cover of A inX, **Conversely**, meanwhile A is N_f −compact with respect to X, it shadows that any N_f −op cover of A in the global sp admits a finite sub-collection $\wedge_1$ of $\wedge$ such that $A \subseteq \bigcup\{U_a: a\in \wedge_1\} \bigcup B^C$ Thus $A\cap B \subseteq \bigcup\{U_a: a\in \wedge_1\}$. Henceforward A∩B is N_f −comp comparative to X.

4. CONCLUSION

In this paper, a new class of sets, which we will call N-feeble-open (N-feeble-O) sets, was introduced and studied in the topological spaces. The study has set up some basic properties of these sets and made their connections with the already defined classes such as feeble-open set, semi-open set and N-semi-open set clear. In particular, the results that are obtained illustrate that the class of N-feeble-O sets gives a desirable extension of the existing generalized concepts of open sets. The following closure and intersection properties of N-feeble-O sets were obtained: The behavior of these sets in subspaces was also considered and related notions such as interior, closure, derived points and adherent points were explored. The results obtained serve as a stepping stone to explore the topological structure brought about by N-feeble-O sets.

Moreover, a new type of compactness of N-feeble-O covers was introduced. This compactness condition was also demonstrated to be more demanding than various related compactness conditions that were examined in the study, such as ordinary compactness, semi-compactness, and feeble compactness. These compactness classes were related and examples were given to show that the converses are not generally true for these classes. The proper behavior of the new introduced compactness concept under subspaces was also explored. The results indicate, among others, that compactness is inherited by closed sets of an N-feeble compact space, and that the compactness is preserved by the appropriate subspace condition. It was also proved that intersections of a compact and a closed set was compact.

As a whole, the results are extensions of the already known concept of generalized open sets and compactness in topology, with the addition of the N-feeble-O sets and the new concept of N-compactness. The relationships and structural properties obtained here serve as a foundation for further study of generalized separation axioms, continuity, compactness and other topological functions related to N-feeble-open sets.

REFERENCES

1. Levin, N. (1963). Semi-open sets and continuity in topological spaces. *The American Mathematical Monthly, 70*, 36–41.
2. Jamil, M. J. (2014). *Utilizing semi-open (S-O) sets in topological spaces* [Master's thesis, University of Baghdad, College of Science].
3. Ahmed, A. S. (2020). *Certain types of spaces and functions by using N-O sets* [Master's thesis, Al-Mustansiriyah University, College of Science].
4. Janković, D. S., & Reilly, I. L. (1985). On semi-separation properties. *Indian Journal of Pure and Applied Mathematics, 16*(9), 957–964.
5. Razzaq, H. M. A., & Ali, H. J. (2023). On NS-O sets in minimal structure spaces. *AIP Conference Proceedings, 2834*(1). AIP Publishing.
6. Malik, A. H., & Ali, H. J. (2025). Feeble b-O set and their application in connecter spaces: A generalization of semi-O and b-O sets. *AIP Conference Proceedings, 3169*(1). AIP Publishing.
7. Ressen, D. A. (2000). *On compact mapping* [Master's thesis, Al-Mustansiriyah University, College of Education, Department of Mathematics].
8. Mirmiran, M. (2000). *A survey on extremally disconnected spaces*. Department of Mathematics, University of Isfahan, Iran, 1–3.
9. Ganster, M., & Reilly, I. (1990). A decomposition of continuity. *Acta Mathematica Hungarica, 56*(3–4), 299–301.
10. Gupta, B. D. (2011). *Topology (with complete metric analysis)* (6th ed.). Kedar Nath Ram Nath.
11. Dorsett, C. (1981). Semi-compactness, semi-separation axioms, and product spaces. *Bulletin of the Malaysian Mathematical Sciences Society, 4*(1), 21–28.
12. Al-Obaidi, J. M., & Al-Azawi, S. N. (2008). On feebly continuous functions and feebly compact spaces. *Diyala Journal for Human Research, 1*(29).
13. Dontchev, J. (1996). Contra-continuous functions and strongly S-closed spaces. *International Journal of Mathematics and Mathematical Sciences, 19*(2), 303–310.
14. Noiri, T., & Popa, V. (2010). The unified theory of certain types of generalizations of Lindelöf spaces. *Demonstratio Mathematica, 43*(1).