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# The Coz-filters correspondence with ideals of C^*(X) (resp. C) on compact spaces

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Abstract

This paper looks for the between filter-ideal connection of $C^*$ (resp. $C$), on compact Tychonoff space ($e.g. \beta X$). The main matter of this work are cozero-set-filters connection and the prime ideal of $C^*(X)$. examining coz-filters property on compact spaces (e.g. Stone–Čech compactification), we found a 1-1 relationship betwixt prime coz-filters and points of $X$, as long as analogue of the duality $p \mapsto M_p$ and maximal z-filter in $C(X)$, and examine under which condition maximal coz-filters on $\beta X$ correspond to points in $C^*(X)$. Confirming that the ideal-filters structures, founded in $C(X)$, stay same in compact setting of $C^*(X)$.

Keywords

Fixed ideal,
Cozero-set,
Coz-filter,
Prime ideal,
z-ideal,
Compact space,
Minimal ideal

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## 1. INTRODUCTION.

Consider here, our space be completely regular Tychonoff space with Stone–Čech compactification $\beta X$, with two rings $C(X)$ and $C^*(X)$, ring continuous and bounded continuous functions for real-valued on X, $C^*(X)$ is subring of $C(X)$, in a compact space $X$, continuous functions are bounded, implies $C(X)$ and $C^*(X)$ are iso. For standard terms follow (Gillman & Jerison, 1960) and (Bhattacharjee & Drees, 2017), and for information on rings and ideals, see (Gillman & Henriksen, 1956), and multiplicative sets, see (Atiyah & MacDonald, 1969) and (Lang, 2002).

The study of the rings $C(X)$ and $C^*(X)$ investigated deeply (Gillman & Jerison, 1960) authors studied in deep relationship between the z-filter of $X$ and ideal structure of this function in the rings. The main correspondence of this theory is $Z$, which associates ideal $I$ of $C(X)$ a z-filter $Z[I]$ on $X$, the detail is in [(Gillman & Jerison, 1960), p: 26–27], we used many primary definitions which are necessary in this work. For further image refer to (Atiyah & MacDonald, 1969), (Fine et al., 1966), (Gillman & Jerison, 1960), (Huckaba, 1988), and (Kaplansky, 1974), coz-filter-prime ideals correspondence was explored in (Bhattacharjee & Drees, 2017) they tested some properties of coz-ultrafilters and its connection with minimal prime ideals of $C(X)$.

The rings located between $C^*(X)$ and $C(X)$ have also attracted attention. In (Plank, 1969) presented a detailed of maximal ideals of such rings, any ring $A(X)$ confirm $C^*(X) \subseteq A(X) \subseteq C(X)$. The maps $\mathcal{Z}_A$ and $\mathfrak{Z}_A$ presented in (Redlin & Watson, 1987) and used in (Redlin & Watson, 1997), (Byun & Watson, 1991), and (Byun et al., 1992) for each ideal $I$ of $A(X)$ and z-filter on X, and so the extension of $E$ and $Z$ to the ring $A(X)$, especially founded in (Panman et al., 2012), that for $A(X) = C^*(X)$ (see (Panman et al., 2012), Corollaries 1.3-2.4). For more type of rings in this structure and relationship between subrings and real compactifications see (Byun et al., 1995) and (Byun et al., 1997). The prime ideal of $C^*(X)$ and $C(X)$ remain in same structure except the bounded function, $P$ is minimal if it includes no least prime z-ideal in fact whenever a maximal ideal contains a prime z-ideal, the collection of all prime ideals contained in that maximal ideal forms a chain under ordered inclusion, this follows from the chain theorem for prime ideals in $C$, furthermore, the structure of prime ideals in $C^*(X)$ has been studied extensively in (Mandelker, 1968). For more on prime ideal and z-ideals of $C^*(X)$ refer to (Mandelker, 1968), (Varadharajan, 2010) and (Aliabad et al., 2010), and some old and new results on ultrafilters are compared in (Comfort, 1977).

We consider the correspondence between coz-filter and points in compact space in $C^*$ (and $C$), first we reform some existing concepts (i.e., $M^{*p}, O^{*p}, B^{*p}\ and\ C^{*p}$) to suitable with the existed space, then show when the coz-filter-ideal connection holds and under which condition don't, inspiring by existing related work in general on $C^*$ and $C$ in §2. And §3 focuses on relation between prime z-ideals and prime coz-filters, we explore relation between points in $X$ and prime z-ideal on $C^*(X)$ and clarifying the required conditions, and showing a bijection between points and coz-filters. Furthermore, inspiring by Mandelker and kohls chain, explore the chains generated by $O^{*p}$ and $M^{*p}$ and investigate some of their properties. The last section focuses on coz-ultrafilters with points. The existing correspondence between maximal ideal (resp. z-ultrafilter) and points in $C$ provides the same relation on $C^*$, which is a spark for a bijection association between coz-ultrafilters and $p$ in $C^*$ by the impact of X being p-space.

Coz-filters structure $C^*(X)$ remain same as $C(X)$ see (Bhattacharjee & Drees, 2017), a coz-filter $\mathcal{F}$ on $X$ is a non-empty collection of $Coz(X)$ that satisfies the finite intersection and subsect property. The $z$-filter-ideal relation in $C(X)$ and coz-filter-ideal relation was explored in (Gillman & Jerison, 1960) and (Bhattacharjee & Drees, 2017), we use those relations for studying the structure of coz-filter-ideal, and find similar relation on $C(X)$ and $C^*(X)$.

If $X$ be a compact then every $f$ on $X$ is bounded implies $C^*(X) = C(X)$ means they are iso (Varadharajan, 2010) Corollary 2.7, and the correspondence in $C^*(X)$ is mediated by $\beta X$, thus $C^*(X)$ is isomorphic to $C(\beta X)$ and their minimal prime ideals connect to each other ((Byun & Watson, 1991), (Sack & Watson, 2013), definition 5.2. (Knox et al., 2009)). Moreover, for any $f, g$ in $C^*(X)$ (same $C$) when $fg = 0$ implies $\mathrm{coz}(f) \cap \mathrm{coz}(g) = \emptyset$ and vise versa.

Recall, in compact space of $C^*$ or $C$, we have the following theorem.

**1.1 Theorem.** (Gillman & Jerison, 1960) Them 4.11. the following are equivalent.

(1) $X$ is compact space.

(2) Let $I$ be any ideal in $C(X)$ then $I$ is fixed (this also means that every z–filter is fixed).

(3) Let $I$ be any ideal in $C^*(X)$ then every ideal in $C^*(X)$ is fixed

(4) Let $M$ be any maximal ideal in $C(X)$ then $M$ is fixed (this also means that every z–ultrafilter is fixed).

(5) Let $M$ be any maximal ideal in $C^*(X)$ then $M$ is fixed.

The results in (Gillman & Jerison, 1960) are well-known as well as the coz-filter-ideal relation on $C(X)$ in (Bhattacharjee & Drees, 2017), but the coz-filter-ideal relation of $C^*$ (resp $C$) on a compact space still uninvestigated. In the following we try to investigate it in detail demonstrate the difference between $z$-filters and coz-filters. When in $C$ (non-compact case) for generating coz-filter on $Coz(X)$ the related ideal need be prime. The key difference here arises since every ideal belonging to any compact space $X$ is fixed (Albalat, 2024) Them 5.11, so to explore the coz-filter-ideal correspondence on a compact space (resp $\beta X$) we deal with fixed ideals rather than free ideals, and this led us to different direction, may the correspond we had in non-compact does not have meaning here because the relation treat with points instead of entire space.

In the following, we first mention some theorems and propositions that we use it within the article.

**1.2 Lemma.** (Gillman & Jerison, 1960) 2.14. $I = Z^{\leftarrow}[\mathcal{F}] \cap C^*(X) = \{f \in C^*(X) | Z(f) \in \mathcal{F}\}$ is an ideal in $C^*(X)$

**1.3 Lemma.** (Gillman & Jerison, 1960) 2D-2 & 5D.

1- (Contrapositive to 2D-2), let $P$ be prime ideal in $C(X)$ (resp $C^*(X)$) and $f, g \in C(X)$ such that $f^2 + g^2 \notin P$, then $f \notin P$ and $g \notin P$.

2- If $f^2 + g^2$ belongs to z-ideal $I$, then $f \in I \,\&\, g \in I$.

3- $f^2 + g^2 \in M^p$ iff $f^2 \in M^p$ or $g^2 \in M^p$ special case of 2D-2 (7.4)

This holds for any prime ideal $P$ in $C^*(X)$, since primes in $C^*(X)$ are absolutely convex, in another word, $0 \le |a| \le |b|$ and $b \in P$ implies $a \in P$, since $f^2 \le f^2 + g^2$ and $g^2 \le f^2 + g^2$, if $f^2 + g^2 \in P$, then $f^2 \in P$ and $g^2 \in P$, as $P$ is prime, $f^2 = f \cdot f \in P \Rightarrow f \in P$, and similarly $g \in P$.

**1.4 Proposition.** The inequality between the zero set of a function and its extension, in general is, $Z(f) \subseteq Z(f^\beta)$ i.e., $cl_{\beta X}\left(Z(f)\right) \subseteq Z(f^\beta)$, when for a bounded continuous $f: X \to \mathbb{R}$, the actual formulas are, $Z(f^\beta) = cl_{\beta X}(Z(f))$ for any $f \in C^*(X)$, to show this, since $Z(f) \subseteq Z(f^\beta)$ and $Z(f^\beta)$ is closed then, $cl_{\beta X}(Z(f)) \subseteq Z(f^\beta)$, on another hand if $p \in Z(f^\beta)$ implies $f^\beta(p) = 0$, by continuity and density of $X$ every neighborhood of $p$ meets $Z(f)$, so $p \in cl_{\beta X}(Z(f))$. On another hand, it is obvious $coz(f^\beta) \subseteq coz(f)$, by duality of zero sets, $coz(f^\beta) = \beta X \setminus Z(f^\beta)$ which is $coz(f^\beta) = \beta X \setminus cl_{\beta X}(Z(f))$(i.e., $\beta X \backslash int(Z(f)) = cl(\beta X \backslash z(f))$ or vice versa $cl \to int$), then in general, $coz(f^\beta) \subseteq int_{\beta X}(\beta X \setminus Z(f)) \subseteq int_{\beta X}(coz(f))$. Thus, for bounded functions $Coz(f^\beta) = int_{\beta X}(coz(f))$. Moreover, since $cl_{\beta X}\left(coz(f^\beta)\right) \subseteq Coz(f)$ we conclude, that $Coz(f^\beta) \subseteq int_{\beta X}\left(cl_{\beta X}\left(coz(f^\beta)\right)\right)$.

The main structural fact from (Gillman & Jerison, 1960) is the isomorphism between $C^*(X)$ and $C(\beta X)$ which makes $\beta X$ the natural surrounding space of $C^*(X)$. Also $C^*(X)$ is genuinely different, since $C^*(X) \cong C(\beta X)$ via $f \mapsto f^\beta$, the natural analogues are obtained by replacing $X$ with $\beta X$ throughout but two things break. First one, the cozero sets of $C^*(X)$ as a ring in its own are cozero sets of $\beta X$ (open sets of the form $coz(f^\beta)$), whereas the cozero sets of $C(X)$ are cozero sets of $X$, these are related by $coz(f^\beta) \cap X = coz(f)$ but $coz(f^\beta)$ can contain points at infinity in $\beta X \backslash X$ that $coz(f)$ misses entirely. Second, invertibility in $C^*(X)$ requires $inf_X|f| > 0$ which is strictly stronger than $coz(f) = X$, this means the E-regularity condition may infiltrates definitions in $C^*(X)$ (e.g. $O^{*p}$) in a way different in $C(X)$.

## 2. FIXED IDEALS.

Since our space is compact (completely regular) as we mentioned the ideals are fixed. Following the same approach, next we mention and name some notation also setup some terms, then examinate some properties and investigate their relation with coz-filters in $C(X)$ and $C^*(X)$, under which condition the results comes out true. Throughout, $X$ is a Tychonoff space, let $C^*(X) \hookrightarrow C(X)$ be the inclusion of bounded into continuous functions, and $\beta X$ is well-known, and $p \in \beta X$, for more fixed ideals refer to (Gillman & Jerison, 1960), (Bhattacharjee & Drees, 2017).

Remember that, $M_p = \{f \in C(X): f(p) = 0\}$ is maximal ideal for $p \in X$, and $O_p = \{f \in C: p \in int(z(f))\}$ is z-ideal, with, $O_p \subseteq M_p$, (back to 4.6-4.7 and 4I in (Gillman & Jerison, 1960) for more on $M_p$ and $O_p$). But when $M_p = O_p$, then p is called a P-point or X is a P-space if every element of X is a P-point (see 4L.2 in (Gillman & Jerison, 1960)). Again $M^p = \{f \in C(X): p \in cl_{\beta X} Z(f)\}$, for $p \in \beta\mathrm{X}$ is the maximal ideal of $C(X)$ and $O^p = \{f \in C(X): p \in \mathrm{int}_{\beta X}(\mathrm{cl}_{\beta X}(Z(f)))\}$ the minimal prime of $C(X)$ in fact it is z-ideal see 14.7 (Gillman & Jerison, 1960)), refer to 7.3-7.12 in (Gillman & Jerison, 1960) for more about $M^p, O^p$. In addition, for $p \in \beta X$, the maximal ideal of $C^*(X)$ are $M^{*p} = \{f \in C^*(X): f^\beta(p) = 0\}$ see 7.2 (Gillman & Jerison, 1960). The same pattern, we define the minimal prime of $C^*(X)$ for $p \in \beta X$, $O^{*p} = \left\{f \in C^*(X): p \in int_{\beta X}\left(Z(f^\beta)\right)\right\}$.

**2.1 Theorem.** For any $p \in \beta X$ and $f \in C^*(X)$, $f \in M^{*p}$ if and only if $p \in cl_{\beta X}Z(f)$

Proof. $\Rightarrow$ Suppose $f \in M^{*p}$, so, $f^\beta(p) = 0$ since $f^\beta$ is continuous on $\beta X$ and $(f^\beta)^{-1}(0) = Z(f^\beta)$ is closed in $\beta X$, we have $p \in Z(f^\beta)$. Now $Z(f) = Z(f^\beta) \cap X$, and since $Z(f^\beta) = cl_{\beta X}Z(f)$ in $\beta X$, we get $p \in cl_{\beta X}Z(f)$, actually more directly, $Z(f^\beta) = cl_{\beta X}Z(f)$ because, $Z(f^\beta)$ is closed in $\beta X$ and contains $Z(f)$. If $q \in cl_\beta XZ(f)$ then every neighborhood of $q$ meets $Z(f)$, so by continuity $f^\beta(q) = 0$, hence $q \in Z(f^\beta)$, then $p \in Z(f^\beta) \Leftrightarrow p \in cl_{\beta X}Z(f)$.

$\Leftarrow$ If $p \in cl_{\beta X}Z(f)$, then every neighborhood of $p$ in $\beta X$ meets $Z(f)$, by continuity of $f^\beta$ and $f^\beta|_{Z(f)} = 0$, we get $f^\beta(p) = 0$, so $f \in M^{*p}$.

**2.2 Proposition.** Recall $A_p = \{z(f): p \in z(f)\}$ is a z-filter $p \in X$ as defined in (Bhattacharjee & Drees, 2017), also $A^p = \{Z \in \mathcal{Z}[X]: p \in \mathrm{cl}_{\beta X}Z\}$ $p \in \beta X$ see (6.5(c)-7.3) (Gillman & Jerison, 1960) (i.e., $A^p = \{Z(f): p \in \mathrm{cl}_{\beta X}Z(f)\}$. $\mathrm{A_p}$ and $A^p$ are $z$-ultrafilters since $A_p = Z[M_p]$ and $Z[M^p] = A^p$, in fact $A^p$ is the unique $z$-ultrafilter on $X$ converging to $p$. According to ((Gillman & Jerison, 1960, 2K) we have, $A_p^* = Z[M_P^*]$. Here the asymmetry becomes critical, since $A^{*p} = Z[M^{*p}]$ is z-ultrafilter in $C^*(X)$ ((Gillman & Jerison, 1960, 2K), so by $A^{*p} = Z[M^{*p}] = \{Z(f): f \in C^*(X), f^\beta(p) = 0\}$, equivalently write, $Z_{\beta X}(f^\beta) = \{q \in \beta X: f^\beta(q) = 0\}$, then, $A^{*p} = \{Z(f): p \in Z_{\beta X}(f^\beta)\}$, for $p \in \beta X$ we define, $A^{*p} = \left\{Z(f): f \in C^*, p \in cl_{\beta X}\left(Z(f^\beta)\right)\right\}$, since $Z(f^\beta)$ is already closed in $\beta X$ and $cl_{\beta X}Z(f^\beta) = Z(f^\beta)$ so the closure is unnecessary because it is redundant but harmless, hence we write for $p \in \beta X$, $A^{*p} = \{Z(f): f \in C^*, p \in Z(f^\beta)\}$.

In (Bhattacharjee & Drees, 2017) the authors defined some new terms like ($B^p, C^p$ we mention $C^p$ later), when $B_p = \{coz(f): p \in coz(f)\}$ is prime coz-filter, same way, for $p \in \beta X$, we define, $B^p = \{coz(f): p \in int_{\beta X}(coz(f))\}$, clearly $B^p$ prime coz-filter, for that, $1^\beta = 1$, then $p \in coz(1^\beta) = \beta X$ so $\emptyset \notin B^p$. Let $coz(f), coz(g) \in B^p$ which gives $p \in int_{\beta X}(coz(f)), p \in int_{\beta X}(coz(g))$ it follows that $p \in int_{\beta X}(coz(f)) \cap int_{\beta X}(coz(g)) = int_{\beta X}(coz(fg))$ then $coz(fg) \in B^p$. If $coz(f) \subseteq coz(g)$ and $coz(f) \in B^p$, since $Z(g) \subseteq Z(f)$ so $cl_{\beta X}Z(g) \subseteq cl_{\beta X}Z(f)$. But $p \in int_{\beta X}(coz(f))$ then $p \notin cl_{\beta X} Z(f)$ then $p \notin cl_{\beta X}Z(g)$, so $p \in int_{\beta X}(coz(g))$ implies $coz(g) \in B^p$, then $B^p$ is coz-filter. Note that $coz(f^2 + g^2) = coz(f) \cup coz(g)$, now if $p \in int_{\beta X}(coz(f^2 + g^2)) \Rightarrow p \in int_{\beta X}(coz(f)) \cup int_{\beta X}(coz(g))$ gives that $p \in int_{\beta X}(coz(f))$ or $p \in int_{\beta X}(coz(g))$ then by proposition 1.4, that $p \in Coz(f^\beta)$ or $p \in Coz(g^\beta)$ implies $f^\beta(p) \neq 0$ or $g^\beta(p) \neq 0$, then, $B^p$ is prime.

Recall $B^p = \{coz(f): p \in int_{\beta X}(coz(f))\}$, so from proposition 1.4, $int_{\beta X}(coz(f)) = Coz(f^\beta) = \beta X \setminus Z(f^\beta)$ which is $coz(f^\beta) = \beta X \setminus cl_{\beta X}(Z(f))$, by replacing, we get $B^p = \{co\,z(f): p \notin cl_{\beta X}(Z(f))\}$ clearly it is the complement of $M^p$, thus by duality, we conclude that, for $p \in \beta X$,$B^p = Coz(C\setminus M^p)$ and $Coz^{\leftarrow}(B^p) = C\setminus M^p$

Now let's define the analogues of $B^P$ in $C^*(X)$, for $p \in \beta X$, $B^{*p} = \{coz(f): f \in C^*, p \in coz(f^\beta)\}$, by definition $p \in coz(f^\beta)$ means $f^\beta(p) \neq 0$, also $f^\beta(p) \neq 0$ means precisely $p \in coz(f^\beta)$ so $p \in coz(f^\beta) \Leftrightarrow f^\beta(p) \neq 0$, then it is equivalent to $B^{*p} = \{coz(f): f \in C^*, f^\beta(p) \neq 0\}$ which is $B^{*p} = \{coz(f): f \in C^*(X) \setminus M^{*p}\}$. These are the same condition but in two different ways.

**2.3 Lemma.** Let $p \in \beta X$ then $B^{\ *p}$ is always coz-filter.

Proof. $1 \in C^*(X) \setminus M^{*p}$ and $coz(1^\beta) = \beta X$, then, $\emptyset \notin B^{*p}$. Let $coz(f), coz(g) \in B^{*p}$ so that $f^\beta(p) \neq 0$ and $g^\beta(p) \neq 0$ which gives $p \notin Z(f^\beta) = cl_{\beta X} Z(f)$ and $p \notin cl_{\beta X} Z(g)$, since $co\,z(f^\beta) = \beta X \setminus cl_{\beta X}(Z(f))$ then $p \in coz(f^\beta)$ and $p \in coz(g^\beta)$ it follows that $p \in coz(f^\beta) \cap coz(g^\beta) = coz((fg)^\beta)$, therefore $p \notin Z(fg)^\beta$ means $(fg)^\beta(p) \neq 0$, then $coz(fg) \in B^p$. If $coz(f) \subseteq coz(g)$ and $coz(f) \in B^{*p}$ for some $g \in C^*$, means $f^\beta(p) \neq 0$ implies $p \notin Z(f^\beta) = cl_{\beta X} Z(f)$ consequently, $p \notin Z(g^\beta)$ gives $g^\beta(p) \neq 0$, then $coz(g) \in B^{*p}$.

Furthermore, in $C^*$, by following the same pattern in $C$ (see (Bhattacharjee & Drees, 2017)), we conclude the correspondence betwwen $B^{*p}$ and $M^{*p}$, such that for $p \in \beta X$, $Coz(C^* \backslash M^{*p}) = B^{*p}$ and, $(Coz)^{\leftarrow}(B^{*p}) = (C^* \backslash M^{*p})$.

Notifying that $\mathrm{B_p} \subseteq \mathrm{C_p}$ for all $\mathrm{p} \in \mathrm{X}$ when, $C_p = \{coz(f): p \in cl(coz(f))\}$, in addition it is dual of $O_p$, inspiringly to $C_p$, we define $C^p$ for $p \in \beta X$, if $f \notin O^p \Leftrightarrow p \notin int_{\beta X}(cl_{\beta X} Z(f))$ from proposition 1.4, $Z(f^\beta) = cl_{\beta X}(Z(f))$ then $p \notin int_{\beta X}\left(Z(f^\beta)\right)$ it follows that $p \in \beta X \backslash int_{\beta X}\left(Z(f^\beta)\right) = cl\left(\beta X \backslash z(f^\beta)\right)$ that is $p \in cl_{\beta X}\left(coz(f^\beta)\right)$, but for $p \in \beta X$ in $C(X)$ we need $cl(f)$ instead $f^\beta$as $O^p$ and $M^p$ do, then from proposition 1.4, $int_{\beta X}(coz(f)) = coz(f^\beta)$ therefore $p \in cl_{\beta X}\left(int_{\beta X}(coz(f))\right)$, thus $C^p = \left\{coz(f): p \in cl_{\beta X}\left(int_{\beta X}(coz(f))\right)\right\}$, generally is not a coz-filter. Let $X = \mathbb{R}^2$ , $p = (0,0) \in X \subseteq \beta X$, define, $f(x,y) = max(0,x)$ , so $coz(f) = \{(x,y): x > 0\}$. And $g(x,y) = max(0,-x)$ , so $coz(g) = \{(x,y): x < 0\}$, then $p = (0,0) \in cl_{\mathbb{R}^2}(coz(f))$ and $p \in cl_{\mathbb{R}^2}(coz(g))$ , but $coz(f) \cap coz(g) = \emptyset$, so $p \notin cl_{\beta X}(\emptyset) = \emptyset$ . Hence $C^p$ not a coz-filter.

Similarly, $B^p \subseteq C^p$ for all $\mathrm{p} \in \beta\mathrm{X}$, since $Z(f) \cup coz(f) = X$ and $X$ is dense in $\beta X$, so, $cl_{\beta X}(Z(f)) \cup int_{\beta X}(coz(f)) = \beta X$, then if $p \notin cl_{\beta X}(Z(f))$ it must $p \in int_{\beta X}(coz(f))$, so $coz(f) \in B^p \Rightarrow coz(f) \in C^p$, which is in parallel to $B_p \subseteq C_p$, also mirroring $O^p \subseteq M^p$. A prime ideal $O^p$, in fact is a minimal prime ideal of $C(X)$, as well, $Coz(C \backslash O^p) = C^p$ and $Coz^{\leftarrow}(C^p) = C \backslash O^p$.

And now, we defining analogue of $C^p$ on $C^*(X)$, $C^{*p} = \left\{coz(f): f \in C^*, p \in cl_{\beta X}\left(coz(f^\beta)\right)\right\}$, for $p \in \beta X$. Similarly, observe that for $p \in \beta X$, $Coz(C^* \backslash O^{*p}) = C^{*p}$ and $Coz^{\leftarrow}(C^{*p})) = C^* \backslash \mathrm{O}^{*p}$.

## 3. PRIME Z-IDEAL.

Since the definition of prime ideals does not change anywhere, except elements, mentioning it is not necessary, refer to (Mandelker, 1968), (Varadharajan, 2010) and (Aliabad et al., 2010). In (Bhattacharjee & Drees, 2017) to establish valid $z$-ideal-coz-filters relationship, the authors put a necessary condition that both structures must satisfy the primeness. The key difference is the natural of boundedness property in $C^*$ or $(C)$, and this led us to these differences (fixed ideals), so in this section we try to explore those relations (fixed ideal-coz-filter) under the boundness impact, and see that, still generate chains between minimal prime and maximal prime ideals like it was in Mandelker and Kohls chain.

**3.1 Definition.** A **z-**ideal in $C^*$ (like $C$) is an ideal that includes every function sharing the same maximal ideals as at least one function already in the ideal (Kohls, 1957, p. 30) and (Gillman & Jerison, 1960) 2.7, 4A.5. So, in $C^*$ z-ideals correspond to z-ideals of $C(\beta X)$ by isomorphism $f \mapsto f^{\beta}$, in another word, ideal $I$ of $C^*(X)$ is a z-ideal whenever $z(\hat{f}) = z(\hat{g})$ in $\beta X$ and $g \in I$ then$f \in I$.

**3.2 Proposition.** (Aliabad et al., 2010) (Proposition 1.1) An ideal I in C(X) is a z-ideal ($z^{\circ}$-ideal) if and only if every prime ideal minimal over I is a z-ideal ($z^{\circ}$-ideal).

**Theorem.** (thm1.8 in (Byun & Watson, 1991)) For any z-ideal $\mathrm{I}$ in $\mathrm{C}^*(\mathrm{X})$, the following are equivalent.

(i) $\mathrm{I}$ is prime

(ii) $\mathrm{I}$ contains a prime ideal.

(iii) Fore very $g, h \in C^*(X)$, if $gh = 0$, then $g \in I$ or $h \in I$.

Proof.

$(\mathrm{i}) \rightarrow (\mathrm{ii})$ and $(\mathrm{iii}) \rightarrow (\mathrm{i})$, are clear.

$(\mathrm{ii}) \rightarrow (\mathrm{iii})$. Let $\mathrm{P}$ be a prime ideal contained in $\mathrm{I}$. If $fg = 0$, then $fg \in I$, so either $f$ or $g$ is in $P$ and hence in $I$.

**3.3 Lemma.** (Albalat, 2024) (Lemma 5.12) Let $X$ be compact invoke the relation $p \rightarrow M_p$ therefore it is a 1-1 relationship, means each $p$ in X is tied to one max-ideal in $C(X)$ .

In the following, we explore the scope of the correspondence betwixt prime z-ideals of $C^*(X)$ and points of $X$.

A bijective correspondence between the prime z-ideals of $C^*(X)$ and the points of $X$ would require two conditions to hold simultaneously, that every prime z-ideal already be maximal and that every maximal ideal be fixed, the first condition is really a statement about $\beta X$ not $X$ under the canonical identification $C^*(X) \cong C(\beta X)$ (Gelfand–Naimark theorem), it becomes, $C(Y)$ is von Neumann regular exactly when every prime ideal of $C(Y)$ is maximal, exactly when $Y$ is a P-space, applying this with $Y = \beta X$ shows that $C^*(X)$ is von Neumann regular if and only if $\beta X$ not $X$ is a P-space. Since $\beta X$ is always compact, and a compact (indeed merely pseudocompact) P-space is necessarily finite ((Gillman & Jerison, 1960, 4K) (Levy, 1977), this single condition already forces $\beta X$, and hence $X$ (dense in $\beta X$), to be finite and discrete. A finite space coincides with its own Stone–Čech compactification, so the second condition is then automatic. The two conditions collapse into one, prime z-ideals of $C^*(X)$ correspond bijectively to points of $X$ if and only if $X$ is finite. For infinite $X$ the correspondence fails, and it already fails among the maximal ideals as soon as $X$ is non-compact well before any question about non-maximal primes arises.

What holds unconditionally, for any Tychonoff X, is the classical Gelfand–Kolmogorov correspondence (Gelfand & Kolmogoroff, 1939), under $f \mapsto f^{\beta}$, $C^*(X) \cong C(\beta X)$, and the maximal ideals of $C^*(X)$ are exactly

$$M^{*p} = \{f \in C^*(X) : f^{\beta}(p) = 0\}, p \in \beta X$$

giving a bijection with all of $\beta X$. Restricting to the fixed maximal ideals those with $p = x \in X$ gives

$$M^{*x} = \{f \in C^*(X) : f(x) = 0\} = M_x \cap C^*(X)$$

a special case of the general correspondence between the maximal ideals of $C(X)$ and of

$C^*(X)$ established by (Gillman et al., 1954). Here, and only here, the correspondence with points of $X$ is exact, $x \leftrightarrow M_x$ is a genuine bijection between $X$ and the fixed maximal ideals see (Lemma 3.4). This is where the correspondence with points of $X$ stops, it is a statement about maximal ideals, not about prime ideals generally, and compactness of $X$ does nothing to prevent non-maximal primes from existing; it only guarantees that every maximal ideal is fixed.

Beneath $M_x$ sits the z-ideal of functions vanishing on some neighborhood of $x$,

$$O_x = \{f \in C(X): x \in int_X(Z(f))\}$$

with $O_x \subseteq M_x$, equality holding exactly when $x$ is a P-point of $X$. It is tempting, but incorrect, to call $O_x$ itself a prime ideal. (Henriksen & Jerison, 1965) show that $O_x$ is precisely the intersection of all the minimal prime ideals contained in $M_x$ for compact $X$ this is stated explicitly (Banerjee et al., 2009), the intersection of all minimal primes contained in $M_p$ is exactly the set of functions vanishing on a neighborhood of $p$. At a non-P-point this intersection is generally the meet of many distinct, pairwise incomparable minimal primes, not a single prime ideal. This is the real reason the point-correspondence cannot extend past the maximal ideals, even at one fixed point $x$, the prime z-ideals contained in $M_x$ form a rich poset with $O_x$ at the bottom rather than a single ideal or a simple chain, containing both long chains of primes and large antichains of incomparable minimal primes. In $C[0,1] = C^*[0,1]$, non-maximal prime ideals strictly below $M_0$ genuinely exist a short Zorn's-lemma argument gives one directly, and shows that any such prime sits inside a unique maximal ideal (Pandey, 2016) and one can further exhibit two incomparable such primes $P, Q \subsetneq M_0$ with $P + Q = M_0$, each built from a different free ultrafilter on a sequence converging to $0$.

The same phenomenon recurs at every point of $\beta X$, not only at points of $X$. For $p \in \beta X$ there is, symmetrically, a z-ideal

$$O^{*p} = \left\{f \in C^*(X): p \in int_{\beta X}\left(cl_{\beta X}(Z(f))\right)\right\}$$

contained in $M^{*p}$, again equal to the intersection of the minimal primes below $M^{*p}$, and contained in every prime ideal of $C^*(X)$ that sits inside $M^{*p}$. Passing to z-filters via the standard map $P \mapsto Z[P]$ an order isomorphism onto its image whenever $P$ is a z-ideal, hence in particular for every minimal prime, since minimal primes are always z-ideals shows that every prime z-filter on X contains $Z[O^{*p}]$ for a unique $p \in \beta X$ and is contained in the z-ultrafilter, $A^{*p} = Z[M^{*p}]$ which is z-ultrafilter in $C^*(X)$, from Proposition 2.2. it is the unique $z$-ultrafilter on $\beta X$ converging to $p$ (on this we talk more about it in section 4).

The prime z-ideals of $C^*(X)$ are therefore in order-preserving bijection with the prime z-filters on X not an order-reversing one, and not simply indexed by points of $\beta X$. Consequently, when $X$ is non-compact, the points $p \in \beta X \setminus X$ contribute additional free maximal ideals of $C^*(X)$ with no counterpart among the points of $X$ at all, compounding the earlier failure, the set of prime z-ideals of $C^*(X)$ outstrips the set of points of $X$ both because a single point already carries an entire poset of primes beneath its maximal ideal, and because $\beta X$ itself already outgrows $X$ the moment $X$ fails to be compact.

We will summarize the above explanation for compact X in the following corollary, phrased in terms of $C^*(X)$ to match the notation used throughout the rest of this note. Since $X$ is compact, $X$ is pseudocompact, so $C(X) = C^*(X)$ as rings a continuous function on a compact space is automatically bounded and the relation $M^{*p} = M_p \cap C^*(X)$ established above degenerates to $M^{*p} = M_p$ see 4.7 in (Gillman & Jerison, 1960), there is no proper subring of bounded functions left to restrict to. What follows isolates exactly where compactness is used, and adds the finiteness criterion already established as a fourth equivalent condition.

**3.4** ***Corollary.*** For a compact Hausdorff space $X$. The following are equivalent.

(1) Map $p \mapsto M^{*p}$ is a bijection between $p \in X$ and the set of prime z-ideals of $C^*(X)$.

(2) Any prime ideal of $C^*(X)$ is maximal.

(3) Every point of $X$ is a P-point of $X$ (equivalently $X$ is a P-space).

(4) $X$ is finite (equivalently discrete).

**Proof.**

*(1) ⇔ (2).* By the Gelfand–Kolmogorov theorem (Gelfand & Kolmogoroff, 1939) since $X$ is compact, $p \mapsto M^{*p}$ is unconditionally a bijection between $X$ and the maximal ideals of $C^*(X)$ this is exactly where compactness is used,

(⇒) If $p \mapsto M^{*p}$ bijects $X$ with all prime ideals, then every prime ideal equals some $M^{*p}$, hence is maximal, (2) holds.

(⇐) If every prime ideal is maximal, then {prime ideals} = {maximal ideals}, and since $p \mapsto M^{*p}$ already bijects $X$ with the maximal ideals (compactness), it bijects $X$ with all prime ideals, (1) holds.

*(2) ⇒ (3).* Fix $x \in X$ and let $O^{*x} = \{f \in C^*(X): p \in int_X(Z(f))\}$ (this for any compact space not only $\beta X$), suppose $Q$ is any prime ideal with $Q \subsetneq M^{*x}$ then $Q$ is proper ideal properly contained in the proper ideal $M^{*x}$, so $Q$ cannot be maximal, contradicting (2), Thus no prime ideal is properly contained in $M^{*x}$, the only prime ideal contained in $M^{*x}$ is $M^{*x}$ itself, since $O^{*x}$ equals the intersection of the minimal primes contained in $M^{*x}$ (Henriksen & Jerison, 1965), stated explicitly for compact $X$ in (Banerjee et al., 2009), and the only candidate is $M^{*x}$ itself, we get $O^{*x} = M^{*x}$. By definition (Gillman & Henriksen, 1954), this says $x$ is a P-point. As $x$ was arbitrary, $X$ is a P-space.

*(3) ⇒ (2).* Let $P$ be any prime ideal of $C^*(X)$, every prime ideal of $C^*(X)$ is contained in a unique maximal ideal (Banerjee et al., 2009), and since $X$ is compact that maximal ideal is $M^{*p}$ for a unique $p \in X$ (Gelfand & Kolmogoroff, 1939), from (3) $O^{*p} = M^{*p}$, since $O^{*p}$ is the intersection of the minimal primes in $M^{*p}$, this forces $M^{*p}$ itself to be the only minimal prime contained in it, but $P \subseteq M^{*p}$ contains some minimal prime $P_0$(every prime ideal of a commutative ring contains a minimal prime, by Zorn's lemma), and $P_0 \subseteq M^{*p}$ forces $P_0 = M^{*p}$. So $M^{*p} = P_0 \subseteq P \subseteq M^{*p}$, giving $P = M^{*p}$, thus $P$ is maximal.

*(3) ⇔ (4).* If $X$is finite, it is discrete (compact Hausdorff + finite ⇒every singleton, hence every subset, is closed and open), and a discrete space is trivially a P-space. Conversely, if $X$ is a P-space, then since $X$ is compact, $X$ is finite (Gillman & Jerison, *1960,* 4K) (Levy, 1977) a compact, or even merely pseudocompact, P-space is finite).

There is one important remark and it worth keeping in mind, the compactness can't be dropped from the hypothesis, and it's worth seeing why concretely, for example take $X = \mathbb{N}$

with the discrete topology, clearly it's a P-space (discreteness gives (3) for free, with no finiteness required off the compact case), so (2) holds too, every prime ideal of $C(\mathbb{N}) = \mathbb{R}^{\mathbb{N}}$ is maximal with no finiteness required, since $X$ is not compact, but (1) fails badly, $x \mapsto M_x$ only reaches the fixed maximal ideals, missing the free ones $M^{*p}$ for $p \in \beta\mathbb{N} \setminus \mathbb{N}$, of which there are $2^{\mathfrak{c}}$. This is the same phenomenon as above Paragraph noted, compactness is doing double duty, forcing both, forcing every maximal ideal to be fixed (needed for (1)⇔(2)), and, together with the P-space property, forcing $X$ to be finite (needed for (3)⇔(4)).

**3.5 Proposition.** If $p \in \beta X$ then $C^p$ prime coz-filter if and only if $O^p$ is a prime ideal.

Proof. Let $fg \in O^p$ then $p \in \mathrm{int}_{\beta X}(\mathrm{cl}_{\beta X}(Z(fg)))$ means $p \notin cl_{\beta X}(coz(fg))$ implies $coz(fg) \notin C^p$, which gives $coz(f) \cap coz(g) \notin C^p$, so, $coz(f) \notin C^p$ or $coz(g) \notin C^p$, but, $C^p$ is a coz-filter. Thus $p \notin cl_{\beta X}(coz(f))$ or $p \notin cl_{\beta X}(coz(g))$, then $p \in \mathrm{int}_{\beta X}(\mathrm{cl}_{\beta X}(Z(f)))$ or $p \in \mathrm{int}_{\beta X}(\mathrm{cl}_{\beta X}(Z(g)))$, so $f \in O^p$ or $g \in O^p$, that is $O^p$ is prime ideal.

Conversely It is clear $\emptyset \notin C^p$. Now, let $coz(f), coz(g) \in C^p$, then $p \notin int_{\beta X}\left(cl_{\beta X}(Z(f))\right)$ and $p \notin \mathrm{int}_{\beta X}(\mathrm{cl}_{\beta X}(Z(g)))$, then $f, g \notin O^p$, but $O^p$ is prime therefore $fg \notin O^p$, which gives $p \notin \mathrm{int}_{\beta X}(\mathrm{cl}_{\beta X}(Z(fg)))$, it follows that $p \in cl_{\beta X}(coz(fg)) = cl_{\beta X}(coz(f)) \cap cl_{\beta X}(coz(g))$, so, $C^p$ is closed under intersection. Let $coz(f) \in C^p$ and $coz(g) \in Coz(X)$, with $coz(f) \subseteq coz(g)$, whereas $cl_{\beta X}(coz(f)) \subseteq cl_{\beta X}(coz(g))$ we have that $coz(g) \in C^p$. Hence $C^p$ is prime coz-filter.

**3.6 Proposition.** $C^{*p}$ is coz-filter (prime coz-filter) of $O^{*p}$ on $\beta X$, for $p \in \beta X$ and $f \in C^*$ if and only if $O^{*p}$ is a prime ideal.

Proof. The proof follows same pattern Proposition 3.6. with $p \in \beta X$ and $f \in C^*$.

Since $P^*(f) \to P(f)$ is order preserving isomorphism, then the relation $P^* = P \cap C^*$ holds for prime $P \in C(X)$, see (Gillman & Jerison, 1960) 7K. In the following theorem we clarify the relationship between the ideals $O^{*p}$ and $O^p \cap C^*(X)$, and when the equality holds.

**3.7 Theorem.** In a Tychonoff space, for every space X and every $p \in \beta X$

$$O^{*p} = O^p \cap C^*(X)$$

Proof. If $f \in O^p \cap C^*(X)$ so, $p \in \mathrm{int}_{\beta X}\left(cl_{\beta X}(Z(f))\right)$, choosing an open $W \ni p \in \beta X$ with $W \subseteq \mathrm{cl}_{\beta X}(Z(f))$, then, as $f^\beta$ is continuous and $Z(f^\beta)$ is closed, and $\mathrm{cl}_{\beta X}Z(f) \subseteq Z(f^\beta)$ then we get $W \subseteq Z(f^\beta)$, by continuity of $f^\beta$ and $f^\beta|_X = f$. So $p \in int_{\beta X}\left(Z(f^\beta)\right)$ implies $f \in O^{*p}$. Thus $O^p \cap C^*(X) \subseteq O^{*p}$

If $g \in O^{*p}$, so there exist $W \ni p \in \beta X$ and $W \subseteq Z(g^\beta)$ *such that* $g^\beta \equiv 0$ on $W$, with extending $g$ to $g^\beta$, we get

$$W \cap X \subseteq Z(g) \dots\dots\dots (a)$$

And for every $q \in W$, and every open neighborhood $U$ of $q \in \beta X$, as $U \cap W$ is open in $\beta X$, then $(U \cap W) \cap X \neq \emptyset$ by density, this showing that for every $q \in W$,

$$W \subseteq cl_{\beta X}(W \cap X) \dots\dots\dots (b)$$

We conclude from(a) and (b), $W \subseteq \mathrm{cl}_{\beta X}(W \cap X) \subseteq \mathrm{cl}_{\beta X}(Z(g))$, but $p \in W$ and $W$ is open and $p \in \mathrm{int}_{\beta X}(\mathrm{cl}_{\beta X}(Z(g)))$ then $g \in O^p \cap C^*(X)$, therefore, $O^{*p} \subseteq O^p \cap C^*(X)$. Hence $O^{*p} = O^p \cap C^*(X)$.

The Mandelker structure for $C^*(X)$, (see (Mandelker, 1968), Theorem I), when $p \in \beta X$ then for any maximal ideal $M^{*p}$ of $C^*(X)$ and the unique maximal ideal $M^p$ of $C(X)$ satisfying $M^p \cap C^* \subseteq M^{*p}$, any prime ideal contained in $M^{*p}$ is comparable with $M^p \cap C^*(X)$, specifically for any prime ideal $P \subseteq M^{*p}$ satisfies $M^p \cap C^* \subset P$ if and only if $P$ contains a unit of $C(X)$. Inspiring by this we can now place $O^{*p}$ precisely in this chain relative to $M^p \cap C^*(X)$, we clarifying in the following theorem.

**3.8 Theorem.** For every $p \in \beta X$, with the standard inclusion $O^{*p} \subseteq \mathrm{M}^{*p}$, the following chain of prime ideals in $C^*(X)$ holds, using Mandelker's structure,
$O^{*p} \subseteq O^p \cap C^*(X) \subseteq M^p \cap C^*(X) \subseteq M^{*p}$.

Note that each containment consists of prime ideals.

Proof. First part follows from Theorem 3.8. and by definition $\mathrm{O}^{\mathrm{p}} \subseteq \mathrm{M}^{\mathrm{p}}$, for the second part, observe that from (Gillman & Jerison, 1960) 7J-2, In $C^*$, $M^{*p}$ is the unique maximal ideal that $O^p \cap C^* \subseteq M^{*p}$, then if there exist a prime $P \subset M^{*p}$ then $O^p \cap C^* \subset P$. Similarly let $f \in \mathrm{M}^{\mathrm{p}} \cap C^*(X)$ means $Z(f) \in \mathrm{A}^{\mathrm{p}}$, so $p \in \mathrm{cl}_{\beta X}(Z(f)) \subseteq Z(f^\beta)$, giving $f^\beta(p) = 0$, that is $f \in \mathrm{M}^{*\mathrm{p}}$. when by definition of $M^p, O^p$ it is obvious $O^p \cap C^*(X) \subseteq M^p \cap C^*(X)$ hold.

**3.9 Theorem.** Let $X$ be a compact space then for $p \in \beta X$ then the following holds,

I. $\mathrm{M}^{\mathrm{p}} \cap C^*(X) = \mathrm{M}^{*\mathrm{p}}$ if and only if $p \in \upsilon X$.

II. $\mathrm{O}^{*\mathrm{p}} = \mathrm{M}^{\mathrm{p}} \cap C^*(X)$ if and only if $p$ is a P-point of $\beta X$

III. $O^{*p} = \mathrm{M}^{*\mathrm{p}}$ if and only if $p \in \upsilon X$ and $p$ is a P-point of $\beta X$

Proof (I). Since $X$ is compact, then, $X = \beta X = \upsilon X$, now let $p \in \upsilon X$ then by (Gillman & Jerison, 1960) theorem 5.8 the maximal ideal $\mathrm{M}^{\mathrm{p}}$ is real, then by (Gillman & Jerison, 1960) 7.9(c) $M^{*p} = M^p \cap C^*$.

Conversely, if $p \notin \upsilon X$, then $C(X)/\mathrm{M}^{\mathrm{p}}$ is a non-Archimedean (hyper-real) field, and there exist $f \in C^*(X)$ with $[f]_{\mathrm{M}^{\mathrm{p}}}$ infinitesimal and nonzero in $C(X)/M^p$, such $f$satisfies $f^\beta(p) = 0$ (i.e., $f \in \mathrm{M}^{*\mathrm{p}}$) but $f \notin \mathrm{M}^{\mathrm{p}}$ (as $f \not\equiv 0 \pmod{M^p}$ in the hyper-real sense), giving a strict inclusion.

(II) Let $p \in \beta X$, in $C(X)$, $O_p = M_p$ for $p \in X$ (4L-J), so, contract it to $C^*(X)$ and consider theorem 3.9. chain, $O^{*p} \subseteq O^p \cap C^*(X) \subseteq M^p \cap C^*(X) \subseteq M^{*p}$, now applying then from theorem 3.8. and theorem 3.10-I gives that $O^{*p} = O^p \cap C^*(X) \subseteq M^p \cap C^*(X) = M^{*p}$, by (Gillman & Jerison, 1960) 7L-2 $\mathrm{O}^{\mathrm{p}} = \mathrm{M}^{\mathrm{p}}$ whereas $p$ is a P-point of $\beta X$, thus $O^{*p} = M^p \cap C^*(X)$.

(III) from (I and II) follows that $O^{*p} = \mathrm{M}^{*\mathrm{p}}$.

Those three conditions, $p \in X$, $p \in \upsilon X$, and $p$ a P-point each collapse a different step in this chain, but it fully collapses when $\mathrm{O}^{*\mathrm{p}} = M^{*p}$ because it requires both $p \in \upsilon X$ and $p$ a P-point together.

From Kohls' chain theorem (Kohls, 1958), for a nonmaximal prime ideal $P \in C(X)$, all prime ideals containing $P$ form a chain under inclusion, in another word, if $P \subsetneq Q$ are prime ideals in $C(X)$ with $P$ nonmaximal then $\mathrm{R} \subseteq Q$ or $Q \subseteq R$ for any prime $\mathrm{R}$ with $P \subseteq \mathrm{R} \subseteq Q$, also parallel to (Gillman & Jerison, 1960) §14.13-14.12, if $P$ and $Q$ are two prime ideals with $\mathrm{P}_{\mathrm{z}} \subsetneq \mathrm{Q}_{\mathrm{z}}$ (where $\mathrm{P}_{\mathrm{z}}$, $\mathrm{Q}_{\mathrm{z}}$ denote their respective $z$-ideal extensions), then every prime ideal $J$ between $\mathrm{P}_{\mathrm{z}}$ and $\mathrm{Q}_{\mathrm{z}}$, with $\mathrm{P}_{\mathrm{z}} \subsetneq J \subsetneq \mathrm{Q}_{\mathrm{z}}$ is comparable to both. And Mandelker Positioning of $\mathrm{O}^{*\mathrm{p}}$ states every prime $P \subseteq \mathrm{M}^{*\mathrm{p}}$ is comparable to $M^p \cap C^*(X)$, see (Mandelker, 1968), Theorem I, therefore in our context if $\mathrm{O}^{*\mathrm{p}} \subsetneq \mathrm{M}^{*\mathrm{p}}$ which holds whenever when $p$ is not a P-point, then every prime laying between $\mathrm{O}^{*\mathrm{p}}$ and $\mathrm{M}^{*\mathrm{p}}$ is comparable with both, and this convincing the claim that the primes below $\mathrm{M}^{*\mathrm{p}}$ form a chain, that makes $\mathrm{O}^{*\mathrm{p}}$ the unique minimal prime ideal contained in $\mathrm{M}^{*\mathrm{p}}$.

Moreover, Theorem 3.9 gives the sharper conclusion, that $\mathrm{O}^{*\mathrm{p}}$ is comparable to $\mathrm{M}^{\mathrm{p}} \cap C^*(X)$ with $O^{*p} \subseteq M^p \cap C^*(X)$, so that in general $\mathrm{O}^{*\mathrm{p}}$ lies strictly below $\mathrm{M}^{\mathrm{p}} \cap C^*(X)$, since $\mathrm{O}^{*\mathrm{p}}$ consists of bounded functions whose $\beta X$-extension *vanishes on a neighborhood* of $p$, and $\mathrm{M}^{\mathrm{p}} \cap C^*(X)$ consists of bounded functions in $\mathrm{M}^{\mathrm{p}}$ (i.e., bounded $f \in C(X)$ with $p \in \mathrm{cl}_{\beta X} Z(f)$). Therefore, under Mandelker's comparability theorem (Mandelker, 1968) Theorem I, the ideal $M^p \cap C^*(X)$ is a distinguished element in the chain of prime ideals between $O^{*p}$ and $M^{*p}$, every prime $P$ with $\mathrm{O}^{*\mathrm{p}} \subseteq P \subseteq M^{*p}$ satisfies
*either* $P \subseteq \mathrm{M}^{\mathrm{p}} \cap C^*(X)$ or $\mathrm{M}^{\mathrm{p}} \cap C^*(X) \subseteq P$,

In particular $\mathrm{O}^{*\mathrm{p}}$ is the minimum of this chain (the intersection of all such primes), so,

$$O^{*p} = \cap\{P \text{ prime in } C^*(X) \colon \mathrm{O}^{*\mathrm{p}} \subseteq P \subseteq \mathrm{M}^{*\mathrm{p}}\}.$$

This is the $C^*$-analogue of the classical (Kohls, 1958) (Gillman & Jerison, 1960) §14 is result in $C(X)$ that, $O^p = \cap\{P \text{ prime} \colon P \subseteq M^p\}$

Then the gap $\mathrm{M}^{\mathrm{p}} \cap C^*(X) \setminus O^{*p}$ is populated by bounded functions vanishing along $p$ but not on a $\beta X$-neighborhood, this gap is empty iff $p$ is P-point of $\beta X$ and every bounded function near $p$ has $Z(f^\beta)$ open again Pseudocompactness, or the local analogue,

Now we turn our attention to coz-filters, and explore its connection with ideals and points, according to information's above we conclude the following correspondence on compact spaces.

**3.10 Lemma**. consider a z-ideal $I$ with $h \in I$ and $Z(h) \subseteq Z(k)$, implies $k \in I$.

Proof. clearly $hk \in I$ for any $h, k$ in $I$, so, $Z(hk) = Z(h) \cup Z(k) = Z(k)$ by hypothesis $Z(h) \subseteq Z(k)$, but $I$ is z-ideal and $Z(hk) = Z(k)$, so, $k \in I$.

**3.11 Theorem.** Let $X$ be compact Hausdorff. Then.

1. There is an order-reversing bijection $\Phi(P) = \mathrm{Coz}(P^c) = \{\mathrm{coz}(f) \colon f \notin P\}$, with inverse $\Psi(F) = \{f \in C(X) \colon \mathrm{coz}(f) \notin F\}$, between the prime z-ideals of $C^*(X)$ all of which are fixed, since $X$ is compact and the prime coz-filters on $X$.
2. Under this correspondence, $p \leftrightarrow B_p := \Phi(M_p) = \{\mathrm{coz}(f) \colon f(p) \neq 0\}$ is a bijection between $p \in X$ and the set of *all* prime coz-filters if and only if every point of $X$ is a P-point (equivalently, $X$ is a P-space, equivalently, $X$ is finite).

Proof.

Since $X$ is compact, $C^*(X) = C(X)$ and $\beta X = X$, so every maximal ideal is fixed, $M^{*p} = M_p$, $p \in X$. Every prime ideal $P$ is contained in some maximal ideal $M_p$, so every $f \in P$ vanishes at $p$, hence $p$ is a common zero of $P$, and $P$ is fixed. So fixed prime z-ideals = all prime z-ideals here.

*(1).* $\Phi(P)$ is a *coz*-filter, $\emptyset \notin \Phi(P)$ since $0 \in P$, $X \in \Phi(P)$ since units are never in $P$, so $\Phi(P) \neq \emptyset$. If $f, g \notin P$ then from primeness $fg \notin P$, so, $coz(f) \cap coz(g) = coz(fg) \in \Phi(P)$. And if $coz(f) \in \Phi(P)$ with $coz(f) \subseteq coz(g)$, then lemma 3.11. gives $g \notin P$, so $coz(g) \in \Phi(P)$, then, $\Phi(P)$ is coz-filter. $\Phi(P)$ is prime. If $coz(f) \cup coz(g) = \mathrm{coz}(f^2 + g^2) \in \Phi(P)$ then $f^2 + g^2 \notin P$, by lemma1.9 gives $f \notin P$ or $g \notin P$, i.e. $coz(f) \in \Phi(P)$or $coz(g) \in \Phi(P)$.

$\Psi(\mathcal{F})$ is a prime z-ideal for prime coz-filter $\mathcal{F}$, let $\mathcal{F}$ be a prime coz-filter put $P = \Psi(\mathcal{F})$, if $f, g \in P$ then $coz(f), coz(g) \notin \mathcal{F}$ and $\mathcal{F}$ is prime, so $coz(f) \cup coz(g) \notin \mathcal{F}$ since $coz(f - g) \subseteq coz(f) \cup coz(g)$ is in $\mathcal{F}$ follows $coz(f - g) \notin \mathcal{F}$, thus $f - g \in P$. Let $f \in P$ and $h \in C(X)$ then $coz(fh) \subseteq coz(f) \notin \mathcal{F}$, so $fh \in P$ implies $P$ is ideal. It is obvious $coz(1) = X \in \mathcal{F}$, so, $1 \notin P$. If $fg \in P$, then $coz(f) \cap coz(g) = coz(fg) \notin \mathcal{F}$ but $\mathcal{F}$ is a filter so that $coz(f) \notin \mathcal{F}$ or $coz(g) \notin \mathcal{F}$ in another word $f \in P$ or $g \in P$, so, $P$ is prime. If $z(f) = z(g)$ and $f \in P$, then $coz(f) = coz(g) \notin \mathcal{F}$, so $g \in P$, thus $P$ is z-ideal.

$\Phi, \Psi$ are mutually inverse, $\text{Coz}(\Psi(\mathcal{F})^c) = \{coz(f): f \notin \Psi(\mathcal{F})\} = \{\text{coz}(f):\text{coz}(f) \in \mathcal{F}\} = \mathcal{F}$ since $\mathcal{F} \subseteq \text{Coz}(X)$, and for prime z-ideal $P$, $f \in \Psi(\Phi(P))$ iff no $g \notin P$ shares $f$'s cozero-set (i.e., $\Psi(\Phi(P)) = \{f: coz(f) \notin \Phi(P)\} = \{f: coz(f) = coz(g) for\ all\ g \notin P\}$) true exactly when $f \in P$, by lemma 3.11. applied to $Z(f) = Z(g)$. So $\Psi(\Phi(P)) = P$. Order-reversing, $P \subseteq Q \Rightarrow \{f: f \notin Q\} \subseteq \{f: f \notin P\} \Rightarrow \Phi(Q) \subseteq \Phi(P)$.

*(2).* By first part, $\Phi$ is *always* a bijection between all prime z-ideals of $C^*(X)$ and all prime coz-filters on $X$, with $\Phi(M_p) = B_p$. Composing, $p \mapsto B_p$ is a bijection between $p \in X$ and all prime coz-filters if and only if $p \mapsto M_p$ is a bijection between $p \in X$ and all prime z-ideals of $C^*(X)$ which, by corollary 3.5. holds if and only if every point of $X$ is a P-point, equivalently $X$ is a P-space, equivalently $X$ is finite

For sake of contradiction suppose some point is not a P-point, then there exists a non-maximal prime z-ideal $P$ with $O_p \subsetneq P \subsetneq M_p$ and its image $\Phi(P)$ is a prime coz-filter strictly containing $B_p$ (the order is reversed: $P \subsetneq M_p \Rightarrow \Phi(P) \supsetneq B_p$), thus there are strictly more prime coz-filters than points, for compact $X$, furthermore, a compact P-space contains no countably infinite subset, in ZFC this implies the space is finite, thus for compact $X$, the bijection $p \leftrightarrow B_p$ holds iff $X$ is a finite discrete space.

the last thing we want to point out in this section is that, same relation (3.12-2) can be found for $p \mapsto B^p$ and $p \mapsto B^{*p}$ for $p \in \beta X$, its proof is following the definition of $B^p$ and $B^{*p}$, theorem 3.12, corollary 3.5 and §2, we left it to the interested reader.

## 4. COZ-ULTERAFILTER.

In this section we pay our attention to the analogue of max z-filter (coz-ultrafilter), the relation max coz-filter-minimal prime ideal in $C(X)$ have been investigated in (Bhattacharjee & Drees, 2017), they explored several conditions and theorems on it, as well as the z-ultrafilter explored in (Gillman & Jerison, 1960), for more on ultrafilters refer to (Comfort, 1977), in fact, the maximal ideals in $C(X)$ correspond in a natural way to the points of $\beta X$ (Byun & Watson, 1991), in a very different way, it turns out that the maximal ideal in $C^*(X)$ are also in 1-1 correspondence with the points of $\beta X$.

**4.1 Lemma.** Maximal ideals in $C^*(X)$ correspond to points in $\beta X$ in a natural way and are not necessarily $z$-ideals, unlike in $C(X)$. (See Chapter 6 in (Gillman & Jerison, 1960), and (Redlin & Watson, 1987).

Proof. (a) From Points $p$ in $\beta X$ to Maximal Ideals, for each $p \in \beta X$ define,

$$M_p = \{f \in C^*(X) | f^\beta(p) = 0\}$$

When $f^\beta$ is the extension of $f$ to $\beta X$, then $M_p$ is maximal ideal because the evaluation map $ev_p: C^*(X) \to R, f \mapsto f^\beta(p)$ is surjective ring homomorphism with kernel $M_p$, that is $C^*(X)/M_p \cong \mathbb{R}$, hance, $M_p$ is maximal ideal.

(b) From Maximal Ideals to Points in $\beta X$, let $M \subset C^*(X)$ be a maximal ideal, we associate a point $p \in \beta X$ to $M$ as follows. Embed $C^*(X)$ into $C(\beta X)$ via the extension map $f \mapsto f^\beta$ and extend $M$ to an Ideal $M^\beta \subset C(\beta X)$ by taking the closure in $C(\beta X), M^\beta = \{g \in C(\beta X) | g = f^\beta \; for \; some \; f \in M\}$

Since $\beta X$ is compact, every maximal ideal in $C(\beta X)$ is of the form:

$$m_p = \{g \in C(\beta X) | g(p) = 0\}$$

for some $p \in \beta X$, by maximality $\widehat{M} = m_p$ for some $p \in \beta X$, and thus:

$$M = \{f \in C^*(X) | f^\beta(p) = 0\} = M_p$$

If $M_p = M_q$, then for all $f \in C^*(X)$, $f^\beta(p) = 0 \iff f^\beta(q) = 0$, since $C(\beta X)$ separates points, $p = q$, therefore every maximal ideal $M$ in $C^*(X)$ is of the form $M_p$ for some $p \in \beta X$, establishing the bijection.

**4.2 Lemma.** (Kohls, 1957) (Gelfand & Kolmogoroff, 1939) (Lemma 2.1.) Let $p \in \beta X$, then $M^p$ is maximal ideal of $C(X)$, conversely, for every maximal ideal $M$ of $C(X)$ there exist a unique $p \in \beta X$ Such that $M = M^p$.

This means if $M \in C(X)$, $f \in M^p$ that's $f^\beta(p) = 0$, but converse is not true in general

**4.3 Corollary.** (Byun & Watson, 1991) (Corollary 1.9.) each prime ideal $P \in C^*(X)$ is contained in a unique maximal ideal.

Proof. Let $P \in C^*(X)$ be prime ideal contained in two different maximal ideals M and N, hence $P$ is prime by theorem 1.8. $M \cap N$ is z-ideal and containing $P$. But in any commutative ring, intersection of distinct prime ideals is never prime, see theorem 2.10. in (Henriksen & Jerison, 1965).

The ideal-filter relation differ in $C^*(X)$, in the theorem 4.11 in (Gillman & Jerison, 1960) the authors established a completely different correspondence, it is well-known when $X$ is compact every ideal is fixed (in particular every maximal ideal) in $C^*(X)$ and $C(X) = C^*(X)$, and maximal ideals are in a bijection correspondence with points of $X$ via the map $p \mapsto M_p = \{f \in C(X): f(p) = 0\}$, in another word, and by reading the sources carefully, we conclude separate and compatible correspondences, which must not be conflated, which are,

i. *coz*-ultrafilters ↔ minimal prime ideals of *C(X)* (Bhattacharjee & Drees, 2017, Thm. 2.11).

ii. (Gillman & Jerison, 1960, §2K) If $M$ is a maximal ideal in $C^*(X)$, then $Z[M]$ is a *z*-ultrafilter and this is the duality of *z*-ultrafilters ↔ maximal ideals of $C(X) \leftrightarrow points \; of \; X$ (Gillman & Jerison, 1960, Thm2.5 & Thm. 4.11), which entirely separate from the correspondence in (i).

In general, these two ideal types are distinct, every minimal prime ideal is contained in a maximal ideal, but they coincide only under special conditions (e.g., on a P-space, where $B_p$ is a *coz*-ultrafilter for every $p \in X$, per Theorem 2.12 of (Bhattacharjee & Drees, 2017). The confusion arises from substituting $C^*(X)$ for $C(X)$ in the *z*-ultrafilter-maximal ideal correspondence, in $C(X)$ for compact space *z*-ultrafilters (resp Maximal ideals) correspond to points in $\beta X$, there is a one-one correspondence among the z-ultrafilters on X and the points of $\beta X$, each z-ultrafilter converging to its corresponding point (Gillman & Jerison, 1960, 6.5.), and in fixed case $p \in X$ the unique z-ultrafilter converging to $p$ is $A_p$, means distinct z-ultrafilters can't have a common cluster point, again any z-ultrafilter containing a z-filter converging to $p$ also converges to $p$, thus if $\mathcal{F}$ is a z-filter converging to $p$, then $A_p$ is the unique z-ultrafilter containing $\mathcal{F}$ (Gillman & Jerison, 1960, 3.18).

Same pattern in $C^*(X)$, we can't claim that coz-ultrafilters correspond directly to minimal prime ideals likewise in (Bhattacharjee & Drees, 2017), instead we should say, fixed ideals of $C^*(X)$ generate coz-filters that sit inside unique coz-ultrafilters, with equality precisely in the compact case (fixed ideal), in another word for $p \in \beta X$ the maximal ideal $M^{*p}$ of $C^*(X)$ generate a coz-filter $Coz[(M^{*p})^c] = \mathcal{F}$ and is equal to coz-ultrafilter $B^p$ iff $p$ is p-point, recall that point $p \in \beta X$ corresponds to fixed maximal ideal $M^{*p}$ of $C^*(X)$ (Gillman & Jerison, 1960, §6.6(b), §4.9), and point $p \in X$ (when $X$ is compact) corresponds to fixed maximal ideal $M^p = M^{*p}$ (Gillman & Jerison, 1960, Thm 4.11), then since $O^p = M^p \Leftrightarrow p$ is p-point $\Leftrightarrow B^p$ is coz-ultrafilter. Thus $B^{*p}$ correspond to a point $p$ in $\beta X$, and in next theorem we show that.

4.4 **Corollary.** (Bhattacharjee & Drees, 2017) a coz-filter $\mathcal{U}$ is max coz-filter iff for every $\mathrm{coz}(f) \notin \mathcal{U}$ and $f \in C^*(X)$ $\exists$ $\mathrm{coz}(g) \in \mathcal{U}$ $s.t.$ $\mathrm{coz}(f) \cap \mathrm{coz}(g) = \emptyset$.

**4.5 Theorem.** Let $p \in \beta X$ and $f \in C^*$ then $B^{*p}$ is a coz-ultrafilter on $X$ if and only if $p$ is a P-point in $\beta X$.

Proof.

$\Rightarrow$ Assume $B^{*p}$ is a coz-ultrafilter we show $p$ is a P-point in $\beta X$, (in another word $M^{*p} = O^{*p}$ in $C(\beta X) \cong C^*(X)$), equivalently (by (Gillman & Jerison, 1960), 4I) we show that for every zero-set $Z \in \beta X$ containing $p$, we have $p \in int_{\beta X}(Z)$. For that let $Z = Z(h) \subseteq \beta X$ with $p \in Z(h)$, where $h \in C(\beta X)$, then $W = \beta X \setminus Z(h) = coz(h) \subseteq \beta X$ such that $p \notin W$, but $C^*(X) \cong C(\beta X)$, we have $h = f^\beta$ for some $f \in C^*(X)$ so $W = coz(f^\beta)$, and since $f^\beta(p) = 0$ we have $coz(f) \notin B^{*p}$ (because $coz(f) = coz(f^\beta) \cap X = W \cap X$, and $p \notin W$ means $f^\beta(p) = 0$). By Corollary 4.4. since $B^{*p}$ is a coz-ultrafilter and $coz(f) \notin B^{*p}$, there exists $coz(g) \in B^{*p}$ such that, $coz(f) \cap coz(g) = \emptyset$

This means $f \cdot g = 0$ on $X$, so $f^\beta \cdot g^\beta = 0$ on $\beta X$. So, $coz(f^\beta) \cap coz(g^\beta) = \emptyset$, since $g^\beta(p) \neq 0$ (as $coz(g) \in B^{*p}$), we have $p \in coz(g^\beta)$, therefore, $p \in coz(g^\beta) \subseteq \beta X \setminus coz(f^\beta) = Z(f^\beta) = Z(h) = Z$, since $coz(g\^{}\beta)$ is open in $\beta X$, we have $p \in int_{\beta X}(Z)$, since this holds for every zero-set $Z$ containing $p$, so $p$ is a P-point in $\beta X$.

$\Leftarrow$ Assume $p$ is a P-point in $\beta X$ to show $B^{*p}$ is a coz-ultrafilter, by Corollary 4.4, let $coz(f) \notin B^{*p}$ for some $f \in C^*(X)$, then $f^\beta(p) = 0$, so $p \in Z(f^\beta)$, but $p$ is a P-point in $\beta X$, gives that $p \in int_{\beta X}(Z(f^\beta))$. Now let $V = int_{\beta X}\left(Z(f^\beta)\right)$ then $V$ is an open neighborhood of $p$ in $\beta X$ with $V \subseteq Z(f^\beta)$, which means $V \cap coz(f^\beta) = \emptyset$, and $\beta X$ is Tychonoff and $p \in V$, so there exists $h \in C(\beta X)$ such that $h(p) = 1$ and $h|_{\beta X \setminus V} = 0$, that is $p \in coz(h)$, so $coz(h) \subseteq V \subseteq Z(f^\beta)$, let $g = h|_X \in C^*(X)$, then $g^\beta = h$, so $g^\beta(p) = h(p) = 1 \neq 0$. Thus $coz(g) \in B^{*p}$, moreover, $coz(g^\beta) \cap coz(f^\beta) = coz(h) \cap coz(f^\beta) \subseteq V \cap coz(f^\beta) = \emptyset$. So $coz(g) \cap coz(f) = \emptyset$ (since $X$ is dense in $\beta X$), then by Corollary 4.4, $B^{*p}$ is a coz-ultrafilter.

**4.6 Theorem.** If $B^{*p}$ is a coz-ultrafilter on $X$, then $B^{*p}$ is a prime coz-filter.

Proof. This follows from Theorem 4.5. and lemma 2.3.

Recall to (Gillman & Henriksen, 1956) (Gillman & Jerison, 1960) (Mandelker, 1968), the set of $Coz(X)$ whose $cl_{\beta X}$ are neighborhoods of $p$, equivalently coz-sets $coz(f) \subseteq X$ such that some $f \in C^*(X)$ with $f^\beta(p) \neq 0$.

**4.7 Theorem.** Let $X$ be a Tychonoff space and $p \in \beta X$, then for a coz-set $coz(g) \in \mathrm{Coz}(X)$ the following are equivalent.

I. $coz(g) \in B^{*p}$

II. $p \in int_{\beta X}\left(cl_{\beta X}\left(coz(g^{\beta})\right)\right)$, that is $cl_{\beta X}\left(coz(g^{\beta})\right)$ is a neighborhood of $p \in \beta X$.

III. $\exists\, f \notin M^{*p}$ and $coz(f^{\beta}) \in \mathrm{Coz}(\beta X)$ with $coz(f^{\beta}) \cap X = coz(g)$ and $inf_{coz(f^{\beta})}|f^{\beta}| > 0$.

$(I) \leftrightarrow (II)$. Let $coz(g) \in B^{*p}$ implies $p \in coz(g^{\beta})$ from proposition 1.4, $p \in int_{\beta X}\left(cl_{\beta X}\left(coz(g^{\beta})\right)\right)$, since $p$ is a point inside it, thus $cl_{\beta X}\left(coz(g^{\beta})\right)$ is a neighborhood of $p \in \beta X$.

$(II) \rightarrow (III)$. Suppose $p \in int_{\beta X}\left(cl_{\beta X}\left(coz(g^{\beta})\right)\right)$ and select an open set $T \ni p$ in $\beta X$ with $T \subseteq cl_{\beta X}\left(coz(g^{\beta})\right)$, but $\beta X$ is compact Hausdorff (which is completely regular), so there exist $k \in C(\beta X)$ with $k(p) = 1$ and $k \equiv 0$ on $\beta X \setminus T$, put $f := k|_X \in C^*(X)$ so $f^{\beta} = k$, also $f^{\beta}(p) = 1 \neq 0$ this giving $f \notin \mathrm{M}^{*\mathrm{p}}$, also put $coz(f^{\beta}) := \left\{|k| > \frac{1}{2}\right\} \in \mathrm{Coz}(\beta X)$ then $p \in coz(f^{\beta})$,$\inf_{coz(f^{\beta})} | f^{\beta} | \geq 1/2 > 0$, and $coz(f^{\beta}) \subseteq T \subseteq cl_{\beta X}\left(coz(g^{\beta})\right)$, and this satisfies $coz(f^{\beta}) \cap X \subseteq cl_{\beta X}\left(coz(g^{\beta})\right) \cap X = coz(g)$ (because $coz(g)$ is open in $X$ and $cl_{\beta X}\left(coz(g^{\beta})\right) \cap X = cl_X\left(coz(g^{\beta})\right) = coz(g)$). But replacing $coz(f^{\beta})$ with $\{h > 1/2\} \cap \mathrm{coz}(h)$ where $h \in C^*(X)$ has $\mathrm{coz}(h) = coz(g)$, so that $coz(f^{\beta}) \cap X = coz(g)$.

$(III) \rightarrow (II)$. Let $f \in C^*(X) \backslash M^{*p}$ and $coz(f^{\beta}) \in Coz(\beta X)$ with $coz(f^{\beta}) \cap X = Coz(g)$ and $inf_{coz(f^{\beta})}|f^{\beta}| \geq \varepsilon > 0$. Since $f \notin M^{*p}$ we have $|f^{\beta}(p)| = \varepsilon_0 > 0$. Set $W = \left\{q \in \beta X : |f^{\beta}(q)| > \frac{\varepsilon_0}{2}\right\}$, open in $\beta X$ with $p \in W$.

For any $q \in W$ and any open $M \ni q \in \beta X$, by density of $X$ in $\beta X$, there exists $x \in M \cap X$. For $M$ sufficiently small, $|f(x)| > \frac{\varepsilon_0}{4} > 0$, so $x \in coz(f) \subseteq cl_{\beta X}\left(coz(g^{\beta})\right)$. Thus every $q \in W$ lies in $cl_{\beta X}\left(coz(g^{\beta})\right)$, giving $W \subseteq cl_{\beta X}\left(coz(g^{\beta})\right)$. Since $p \in W$ and $W$ is open, $p \in int_{\beta X}\left(cl_{\beta X}\left(coz(g^{\beta})\right)\right)$.

If we use the condition $f \notin O^{*p}$ instead of $f \notin M^{*p}$, as explained above, it is incorrect and fails since $f \notin O^{*p}$ is strictly weaker, this admits functions in $M^{*p} \backslash O^{*p}$ that vanish at $p$ and therefore cannot sustain $inf_{coz(f^{\beta})}|f^{\beta}| > 0$ on any neighbourhood $coz(f^{\beta})$ of $p \in \beta X$. For that if $f \in M^{*p}$ means $f^{\beta}(p) = 0$ then for any open $coz(f^{\beta}) \ni p \in \beta X$, continuity of $f^{\beta}$ forces $inf_{coz(f^{\beta})}|f^{\beta}| = 0$ (since the infimum is achieved along any net converging to $p \in coz(f^{\beta})$). Hence no $f \in M^{*p}$ can serve as witness in (iii). This confirms that the condition $f \notin O^{*p}$ is insufficient and replacing $O^{*p}$ by $M^{*p}$ is a necessary and sufficient condition.

## 5. Conclusion.

Here we delved into the association between filters of coz-sets and ideals on a Tychonoff space (compact). We reformed some existing notations to be compatible with our space, then showing that under which condition they satisfying coz-filter conditions, and explore their relation to fixed ideals in compact spaces. Then we delve more deep in subject and investigated the prime z-ideal-point-prime coz-filter correspondence, Through the detailed examination of coz-ultrafilters on $\beta X$, we established a bijective correspondence with the points on $\beta X$ correspondence inspiring the known point-z-ultrafilter-maximal ideal relationship.

Furthermore, the capability of generating chains according to Mandelker's chain and Kohl's chain, we explored the conditions and situations for forming that chain in $C(X)$ and limit it to $C^*(X)$ and considering the relations with intermediate ideals.

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