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# Mathematical Model for the Dynamics of Insurgents-Security Forces Interaction under the Combined Effects of Counseling and Rehabilitation

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### ABSTRACT

Due to the complex underlying roots of Insurgency, it has become a global issue in the 21<sup>st</sup> Century, efforts to curtail insurgency (and to a wider scale, terrorism), in the form of militarized counter-insurgency and counter terrorism come with huge capital budgetary expenditure on the states and concerned parties. In some cases, the states and concerned parties have to source for loans and conditional financial support from foreign powers with its attendant consequences on their economic and socio-cultural ethics and values. Despite these efforts, insurgency and terrorism have not been tamed appropriately, in fact the menace is none decreasing, particularly in Africa. This point to the fact that; sole militarization of counter insurgency interventions is not enough to tame insurgency. In this study, we look into the effect(s) of rehabilitation through counseling on insurgents. The population of insurgents was decoupled into inactive and active insurgents because of the collaborative interactions between the two sub-populations to develop a mathematical model for the dynamics of interactions between security-forces and insurgents under the influence of rehabilitation through counseling as counter-insurgency approaches. In developing the model, the interaction between the security-forces and the insurgents was assumed to be prey-predator like, in which the former is the predator and the latter is the prey or vice-versa, and assumed logistic growth capacity for each of the decoupled insurgent sub-population. The security-forces deployment was assumed to be a function of the insurgents' activities. On the basis of this, we proposed a system of three non-liner ordinary differential equations that describes the interaction between; active, non-active insurgents and security forces. We carried out a qualitative analysis of the model, where we established the equilibrium point solution, positivity and invariant region for the solution for the model. We computed five equilibria for the model namely; trivial, endemic insurgents, inactive-insurgents-free, active-insurgents-free, security-forces-free equilibrium points. The trivial and security-forces-free equilibrium were found to be locally asymptotically unstable. The inactive-insurgents-free and the active-insurgents-free equilibria were proved to be conditionally locally asymptotically stable.

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## INTRODUCTION

INSURGENCY is a struggle for power (*over a political space*) between a state (*occupying power*) and one or more organized popularly based internal challenges [1]. Insurgency is further defined as a violent, armed rebellion by small, lightly armed bands who practice guerrilla warfare from primarily rural base areas against a larger authority. The key descriptive feature of insurgency is its asymmetric nature, small irregular forces face a large, well-equipped, regular military force state adversary. Due to this asymmetry, insurgents avoid large-scale direct battles, opting instead to blend in with the civilian population (mainly in the countryside) where they gradually expand territorial control and military forces.<sup>[4]</sup> Insurgency frequently hinges on control of and collaboration with local populations. Insurgency emerges from many interactions between numerous social, economic, and geographical factors. Adequately accounting for the large number of potentially relevant interactions, and the complex ways in which they operate, is key to creating valuable models of insurgency.

However, this has long been a challenging endeavor, as insurgency imposes specific limitations on the data that could speak to these interactions, that is, quantitative data is limited by the difficulties of systematic collection in war, while qualitative data may include vague or conflicting insights from direct observers [2]. In the work of [3], they opined that; conducting operations against an unconstitutional enemy is a difficult task for the army. This opinion was buttressed by the words of [4], that; government and the insurgents are engaged in an armed conflict governed by violent actions and mutual attrition. According to [5], insurgency seeks to achieve the following:

- Undercutting the ability of the government to provide the population with security and public services.
- Obtaining the active and passive support of the population.
- Provoking the government into committing abuses that drive neutral civilians towards insurgents and solidify the loyalty of insurgents' support.
- Undermining international support for the government, and if possible, gain international recognition and/or support for assistance for the insurgency.

War is one of the many consequences of human

behavior. Human behavior may often be difficult to predict, let alone modelled. Successful counterinsurgency is dependent upon human interactions and the consequences of those interactions. Hence, modelling counterinsurgency campaign to provide insight into the appropriate level of forces and operations, presents a challenging task [6]. The mechanisms driving the nucleation, spread, and dissipation of crime hotspots are poorly understood. As a consequence, the ability of law enforcement agencies to use mapped crime patterns to design crime prevention strategies is severely hampered. We also lack robust expectations about how different policing interventions should impact crime [7]. The military has continually refined the ways in which it measures success in counterinsurgency [6].

For example, in Vietnam, counting the bodies of the enemy was a way that commanders thought could measure the success of the campaign. This, however, proved to be ineffective as the nature of the conflict was further understood [6]. In this context, the rapid spread of radical ideologies has led to a world-wide succession of terrorist attacks in recent years. Understanding how extremists' tendencies germinate, develop, and drive individuals to action is not only important from a cultural standpoint, but also to help formulate response and prevention strategies.

Insurgencies are different from regular armed conflicts in the way they are physically manifested and the role civilians play in them [4]. Common threats in insurgencies are road-side attacks by Improvised Explosive Devices (IEDs) and suicide bombing – mostly used as terrorizing and coercing means against civilians [4]. The strategic interplay between counter terror measures and terror activity is complex [8]. Furthermore, demographic studies, interviews with radicalized subjects, analysis of terrorist databases revealed that the path to radicalization occurs along progressive steps, where age, social context and peer-to-peer exchanges play major roles [9].

To execute terrorist attacks, radicals must efficiently communicate with one another while maintaining secrecy. They are also subject to pressure from counter-terrorism agencies, public opinion and the need for material resources. Similarly, government entities must gauge which intervention methods are most effective. While a complete understanding of the processes that

leads to extremism and violence, and of which deterrents are optimal is still lacking.

Mathematical modelers have contributed to the discourse by using tools from statistical mechanics and applied mathematics to describe existing and novel paradigms, and to propose novel counter-terrorism strategies [9]. Furthermore, the authors in [10] opined that; Mathematical research of terrorism has the potential to inform both scholars and policymakers. In this context, [10] presented the following four results from their thesis, viz; (1) a mathematical model of a terrorist organization focusing on evaluating various counter-terrorism measures and predicting the evolution of terrorist conflicts; (2) a mathematical model of nuclear smuggling, where the adversary is described as a Markov process on a transportation network, and algorithms for positioning sensor arrays on the network; (3) a new formulation of nuclear smuggling that allows fast computation using approximation algorithms with performance guarantees; (4) a model for constructing cascade-resilient networks, with implications for analyzing the structure of terrorist networks, specifically their susceptibility to betrayal. His work made the following specific contributions:

- A general definition of the problem as a multi-objective optimization problem.
- Metrics for efficiency and resilience that work well in various networks, including disconnected topologies.
- Evidence for optimality of star-like topologies.
- Evidence for non-monotonicities in the edge density as a function of cascade risk.
- Evidence that the cellular hierarchical network, such as; the *Francs-tireurs Partisans* (FTP), is suitable for intermediate ranges of risk, and that the 9/11 network would have been easily defeated under a more rigorous security environment.

Insurgency compelled to bear direct, indirect and hidden cost to a nation [11]. For example, In Nepal, a decade-long insurgency and counterinsurgency claimed many lives, destroyed infrastructure, caused internal displacement of people, and migration to India [11].

The work of [12] examined the relationship between violent conflict and childhood wasting in Northeast Nigeria, where residents have been subjected to fighting between the Nigerian government and Boko Haram since

2009. They used two Demographic and Health Surveys from before and after the Boko Haram insurgency started, and used double-difference (difference-in-difference) approach to assess the impact of the conflict on mean weight-for-height z-scores and the likelihood of wasting. The results of their study suggest that if children exposed to the conflict had not been exposed, their mean weight-for height z-score would be 0.49 standard deviations higher ( $p < 0.001$ ) than it is, increasing from  $-0.74$  to  $-0.25$ . Additionally, the likelihood of wasting would be 13 percentage points lower, bringing the proportion down from 23% to 10%. They concluded that; “Descriptive evidence suggests that poor child health outcomes in the conflict areas of Northeast Nigeria may be due to disruptions to social services and increased food insecurity in an already resource poor area. Although other unidentified factors may contribute to conflict and wasting, the findings underscore the importance of appropriate programs and policies to support children in conflict zones” [12].

In the work of [13], they uncovered a simple dynamical pattern that may be used to estimate the escalation rate and timing of fatal attacks. They estimated the time difference between fatal attacks by insurgent groups within individual provinces in both Afghanistan and Iraq, and by terrorist groups operating worldwide, this gave a potent indicator of the later pace of lethal activity.

The main focus of existing research on modeling insurgency and counter insurgency is on organizational analysis and its implications for militant group(s) operational processes and performance. Few studies investigated how differences in network structure led to divergent outcomes with respect to political processes such as militant group infighting, their strategic use of violence, or how politically salient variables affect the evolution of militant cooperative networks [9]. Modeling counterinsurgency to accurately portray relevant data can benefit a tactical commander and staff, this can enable them continue refining their plans and operate within the framework of their doctrine. A model can also help establish reasonable goals for the Army unit conducting operations so as not to overreach their capabilities, but utilize resources in effective ways to achieve the desired end state [6].

In [8], they proposed a dynamic model to depict the interaction between counter terror measures and terror activity. The model generates the following stylized

prognoses:

- i. under conditions of inefficient counter terror measures, terror groups enjoy longer period of activity but only if recruitment into terror groups remains low.
- ii. highly efficient counter terror measures effectively contain terror activity, but only if recruitment remains low. Thus, highly efficient counter terror measures can effectively contain terrorism if recruitment remains restrained. They concluded that the trajectory of the dynamics between counter terror measures and terror activity is heavily altered by recruitment.

The work of [9] reviewed some approaches of the terrorist, using; compartment modeling for populations of increasingly extreme views, continuous time modeling for age-structured radical populations, agent-based models, and game theoretic formulations. They highlighted the useful insights offered by analyzing radicalization and terrorism through quantitative frameworks. Finally, they discussed the role of institutional intervention and the stages at which de-radicalization strategies might be most effective.

The authors in [14] considered the problem of suicide terrorism, government counter terror responses, and the mobilization of recruits in support of the contending parties, and developed a model that enables predictions as to what factors should be emphasized or de-emphasized by the forces protecting society. The article presents a linear mathematical analysis of the logical interrelationships involved in the confrontation and embeds the study within the framework of previous mathematical and empirical work on the subject. They concluded that governments should avoid inflicting collateral damage on the general population in counterterrorist activities and should pursue policies (both in counterterrorist operations and otherwise) that contribute to the political quiescence of the populace.

In another way, [2] designed a computational framework based on Fuzzy Cognitive Maps and Complex Networks. They presented a software solution that fully implements this framework and allows analysts to conduct simulations to better understand the current dynamics of insurgency or test 'what-if' scenarios. Two approaches were presented by [5] to guide analysts in developing models based on their framework, either through a

nuanced reading of the literature, or by aggregating the knowledge of domain experts.

The work of [15] uncovered a simple dynamical pattern that may be used to estimate the escalation rate and timing of fatal attacks. They estimated the time difference between fatal attacks by insurgent groups within individual provinces in both Afghanistan and Iraq, and by terrorist groups operating worldwide. This gave a potent indicator of the later pace of lethal activity. They further used generative mathematical models to interpret certain statistical patterns observed in insurgent, terrorist, and cyber-attacks across different geographic settings including cyberspace. They revealed important mathematical insights underlying a previously identified universal power law exponent that fits various real-world datasets of attack severity associated with insurgent and terrorist activities. They also looked at the timing and frequency of successive attacks, identifying an empirical relation between the initial attack frequency, and the escalation rate of follow-up attacks. On the basis of their analysis, they conjecture that, irrespective of the underlying circumstances and locations, groups of humans tend to "do" insurgency and terrorism in a surprisingly generic and universal way.

Another mathematical model for insurgency (IN) and counterinsurgency (COIN) operations with a large-scale system of differential equations that is connected to a coalition network model was developed by [16]. Their model combines elements of traditional differential equation force-on-force modeling with modern social science modeling of networks, psychological operations (Psycho), and coalition cooperation to inform the tactics and strategies of counterinsurgency decision makers in USA. Their simulations analyzed components of leadership, promotion, recruitment, financial resources, operational techniques, network communications, coalition cooperation, logistics, security, intelligence, infrastructure development, humanitarian aid, and psychological warfare, with the goal of informing today's decision makers of the options available in counterinsurgency tactics, operations, and strategy. The result of their study suggested that; In order to be more effective, the US military must improve its counterinsurgency capabilities and flexibility to match the adaptability of insurgent networks and terror cells.

A model for the connection between the rate of terrorist events and the occurrence of counter-terrorism

interventions in order to examine the relative effectiveness of the interventions could be found in [17]. Using data from the Global Terrorism Database and information on interventions collected by the authors, model results showed that for Indonesia, the Philippines and Thailand, reactive types of interventions, such as arrests, indictments, imprisonments, assassinations and other operational activities indicated the strongest association with the rate of terrorism incidents over time. Maximum effectiveness—as measured by the number of days after an intervention exhibits its full inhibitory effect on the risk of subsequent terrorist attacks—was found to be greatest in Indonesia and the Philippines (11 days and 8 days respectively) and least effective in Thailand (impacting only on the day the intervention occurred). They also examined the number of days after an intervention that the response was able to maintain a high level of effectiveness—17 days in Indonesia, 13 days in the Philippines and one day in Thailand. There were significant differences across these three countries and these results highlight a new approach to conceptualizing the interaction between terrorism and counter-terrorism efforts.

Another mathematical framework based on reaction-diffusion partial differential equations for studying the dynamics of crime hotspots was presented by [7]. The system of equations is based on empirical evidence for how offenders move and mix with potential victims or targets. Their analysis shows that crime hotspots are formed when the enhanced risk of repeat crimes diffuses locally, but not so far as to bind distant crime together. Crime hotspots may form as a result of either supercritical or subcritical bifurcations, the latter is as a result of large spikes in crime that override linearly stable, uniform crime distributions [7]. Their mathematical methods showed that subcritical crime hotspots may be permanently eradicated with police suppression, whereas supercritical hotspots are displaced following a characteristic spatial pattern. The results thus provide a mechanistic explanation for recent failures to observe crime displacement in experimental field tests of hotspot policing [7].

Another dynamical model of terrorism was developed by [18]. They considered a population in a given region as being made up of three primary components: terrorists, those susceptible to both terrorist and pacifist propaganda, and non-susceptible or pacifists, and incorporated (1) the effects of both direct military/police

intervention to reduce the terrorist population, (2) nonviolent and persuasive intervention to influence the susceptible to become pacifists, and studied the dynamical behavior of these three populations using the model. Their work proposes a new paradigm for studying terrorism, and looks at the long-term dynamical evolution in time of these three population components when such interventions are carried out. Many important features (some intuitive, others not nearly so) of the nature of terrorism emerge from the dynamical model proposed, these features lead to several important policy implications for the management of terrorism.

Similarly, in the work of [19] constructed a Mathematical model of terrorism with military strategies and rehabilitation of terrorists. The model was developed with a view to control the spread of terrorist ideologies in the society. The population was divided into six compartments: Susceptible  $S(t)$ , Moderate  $I(t)$ , Terrorist  $T(t)$ , Leader  $TL(t)$  and Foot-soldiers  $TS(t)$  and Detention facilities  $QT(t)$ . The result of the sensitivity analysis shows that the most sensitive parameters are the recruitment pool of the terrorist from susceptible to moderate ( $\beta_1$ ) and terrorist move to detention facilities due to counter terrorist activities ( $b$ ). The least parameter is the probability at which terrorist become militancy leaders ( $k\alpha$ ). ( $\beta_1$ ) and ( $b$ ) are parameters of counter-terrorist that need to be targeted. The findings show that the military/dialogue strategies are to be used while military strategies alone should not be used if the number of terrorists is below a certain basic reproduction number.

A computer simulation model to yield valuable insights into dynamics of insurgency evolution, determine insurgency mitigating conditions and estimated cost of the insurgency was developed by [11]. The main hypothesis of their research is that the lack of understanding of the dynamics of insurgency development and mitigation has contributed to the cost of the conflict. Their study aims to apply system Dynamics (SD) methodology with conflict transformation theory to examine the development, management and cost of conflict. The study finds that full reliance on armed solution might not be a good answer for any conflict. The cost of armed conflict might always be greater than its benefit. The core insight gained from their study, is that; the most powerful instrument that shapes the future of peace and security, is the self-examination of costs and dynamics of the insurgency. On the basis of their results, they made the following recommendations:

First, the insurgency should not be thought of in military terms only, but it should be scrutinized in light of the national strategy and the implementation of state capacity elements. Military, diplomatic, socio-economic, governance and legislative efforts all must be synchronized and united toward achieving the common principal objective, the defeat of the insurgent and its underlying causes. Addressing transitional security requires improving the elements of national power including democracy and governance.

Second, the effort for the state must be to gain popular support, which gives legitimacy to the security force operation. It provides the necessary intelligence to locate the insurgent members and removes the recruitment base from the potential insurgent. Insurgents do also rely on public support, without public support to them; the resources needed for their survival and actions are no longer available.

Third, the security force must limit the use of direct action against the insurgent without having good intelligence so as to minimize innocent casualties. The state should focus on moderating their recruitment base by improving state capacity elements such as the governance, employment, legitimacy and the economy.

Fourth, the short and mid-range strategies should be designed to disrupt and control the insurgency while the long-range strategy should focus on ideological support to the insurgent and the culture of violence.

Fifth, Continuous sincere effort must move toward socio-economic-political reform in post-conflict situation to prevent the country being trapped into further conflict.

Sixth, Insurgency, organized crimes and insecurity, should not be limited to the burden of the country concerned, rather recognize the problem of the international community as a whole. The state should emphasize gaining support of the international community for moral and resource support.

Seventh, cost of the conflict tends to be incorrectly perceived and underestimated. Economic losses have long term impacts on the economy; therefore, it should be given proper attention and dedicated policies should be sought to minimize the adverse economic effects.

## MODEL FORMULATIONS

### A. Assumptions of the Model

1. The interaction between the insurgents and security forces is treated as prey-predator interaction, where the prey is the insurgent and the predator is the security forces.
2. The population of the insurgents is partitioned into; Inactive (Passive) and Active Insurgents.
3. The active insurgents have capacity level to attack the security forces.
4. The population of insurgents in the absence of security forces grows logistically with carrying capacity of the habitat.
5. Both the insurgents and security forces die either naturally at constant rate, or as a result of encounter with each other at different constant rates.
6. Insurgents captured alive or surrendered undergo thorough rehabilitation and counseling program.
7. Security forces loyalty to constituted authority remain uncompromised, no security force member defects to insurgency.
8. Thorough screening is applied when recruiting security forces to avoid penetration of insurgents into the security forces for sabotage of services and compromise.
9. Parameters of the model are strictly nonnegative and will be sourced from existing literatures or will be assumed to fit the model were otherwise.

### B. Definitions of Variables and Parameters as Used in the Model

The definitions of variables and parameters for the model are presented in TABLE I.

TABLE I  
UNITS FOR MAGNETIC PROPERTIES

| Variable/Parameter | Definition                                |
|--------------------|---|
| $x(t)$             | Number of inactive insurgents at time $t$ |
| $y(t)$             | Number of active                          |

|            |   |           |  |   |
|------------|---|-----------|--|---|
| $s(t)$     | insurgents at time $t$  |           |  | insurgents due to   |
|            | Number of security forces at time $t$   |           |  | engagement with the security forces   |
| $K$        | Environmental carrying capacity   | $\sigma$  |  | Recruitment rate for the security forces  |
| $r_1$      | Logistic growth rate for the inactive insurgents  |           |  | due to actions of the inactive insurgents   |
| $r_2$      | Logistic growth rate for the active insurgents  | $\pi$     |  | Recruitment rate for the security forces due to actions of the active insurgents    |
| $\phi$     | The initiation rate from the state of being inactive to becoming an active insurgent            | $\varphi$ |  | The constant killing rate of the security forces by the inactive insurgents         |
| $\alpha$   | Rehabilitation rate for the inactive insurgents captured alive by the security forces           |           |  | when they engage with them or through suicide missions                              |
| $\beta$    | Rehabilitation rate for the active insurgents captured alive by the security forces             | $\mu$     |  | The constant killing rate of the security forces by the active insurgents           |
| $\gamma_1$ | The constant killing rate of the inactive insurgents due to engagement with the security forces | $d$       |  | when they engage with them or through suicide missions                              |
| $\gamma_2$ | The constant killing rate of the active   |           |  | The reverse decay constant rate of the security forces in the absence of insurgency |

From the assumptions and the definitions of the variables and parameters of the model, we derived the following model equations:

$$\begin{aligned} \frac{dx(t)}{dt} &= r_1 x(t) \left( 1 - \frac{x(t)}{K} \right) - \alpha x(t) s(t) \\ &\quad - \phi x(t) y(t) - \gamma_1 x(t) s(t) \end{aligned} \quad (1)$$

$$\begin{aligned} \frac{dy(t)}{dt} &= r_2 y(t) \left( 1 - \frac{y(t)}{K} \right) + \phi x(t) y(t) \\ &\quad - \beta y(t) s(t) - \gamma_2 y(t) s(t) \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{ds(t)}{dt} &= \sigma x(t) s(t) + \pi y(t) s(t) \\ &\quad - \varphi x(t) s(t) - \mu y(t) s(t) - ds(t) \end{aligned} \quad (3)$$

Subject to the initial conditions:

$$\begin{aligned} x(0), y(0), s(0) &\geq 0, \\ x(0) + y(0) + s(0) &\leq K, \\ (\pi - \mu) > 0, (\sigma - \varphi) &> 0. \end{aligned} \quad (4)$$

The system (1)–(3) subject to the initial conditions in (4) can be written in vector form as:

$$Y'(x, y, s) = F(x, y, s), \quad (5)$$

$$\text{where } Y' = \begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \\ \frac{ds(t)}{dt} \end{bmatrix}, \text{ and}$$

$$\begin{aligned} F(x, y, s) &= \begin{bmatrix} F_1(x, y, s) \\ F_2(x, y, s) \\ F_3(x, y, s) \end{bmatrix} \\ &= \begin{bmatrix} r_1 x(t) \left( 1 - \frac{x(t)}{K} \right) - \alpha x(t) s(t) \\ -\phi x(t) y(t) - \gamma_1 x(t) s(t) \\ r_2 y(t) \left( 1 - \frac{y(t)}{K} \right) + \phi x(t) y(t) \\ -\beta y(t) s(t) - \gamma_2 y(t) s(t) \\ \sigma x(t) s(t) + \pi y(t) s(t) \\ -\varphi x(t) s(t) - \mu y(t) s(t) - ds(t) \end{bmatrix} \end{aligned}$$

## MODEL ANALYSES

The model analysis begins with basic analyses of the model as follows:

### 1. Region of Absolute Stability

We begin the model analyses by choosing an invariant set  $\Omega$  such that, all the feasible solutions of the model are uniformly bounded in a proper subset of  $\Omega$ . The feasible region

$$\Omega = \left\{ \begin{array}{l} (x(t), y(t), s(t)) \in \mathfrak{R}_+^3 : \\ x(t) + y(t) \leq K, s(t) \leq \frac{K(\sigma + \pi)}{d} \end{array} \right\} \quad (6)$$

is considered for the model analysis. Hence all feasible solutions of the model enter the region  $\Omega$  which is a positively invariant and attracting set. Thus, the model is well-posed and meaningful in the domain of  $\Omega$  and it is sufficient to consider the dynamics of the model described by (1)–(3) and the initial conditions in (4) within the domain of  $\Omega$ .

### 2. Existence and Uniqueness of Solutions

Consider the system  $Y'(x, y, s) = F(x, y, s)$ , describing the dynamics between insurgents and security forces, Suppose that,

$F_1(x, y, s), F_2(x, y, s), F_3(x, y, s)$  and  $\frac{\partial F_1(x, y, s)}{\partial x}, \frac{\partial F_2(x, y, s)}{\partial y}, \frac{\partial F_3(x, y, s)}{\partial z}$  are continuous and bounded functions

defined on a region  $R = \left\{ \begin{array}{l} (x, y, s) = x_0 - \delta < x < x + \delta, \\ y_0 - \delta < y < y_0 + \delta, s_0 - \delta < s < s_0 + \delta \end{array} \right\} \quad (7)$  Then, there exists a number  $\delta_1$  (possibly smaller than

$\delta$ ) so that the solution  $Y = f(x, y, s)$  for the insurgency model exist and is unique for  $x_0 - \delta < x < x + \delta, y_0 - \delta < y < y_0 + \delta, s_0 - \delta < s < s_0 + \delta$ .

**Proof:**

Defining  $\delta$  to be  $\delta = \frac{d}{\pi + \sigma}$ , then, clearly  $F_1(x, y, s), F_2(x, y, s), F_3(x, y, s), \frac{\partial F_1(x, y, s)}{\partial x}, \frac{\partial F_2(x, y, s)}{\partial y}$ , and

$\frac{\partial F_3(x, y, s)}{\partial z}$  are continues on the entire  $R = \left\{ \begin{array}{l} (x, y, s) = x_0 - \delta < x < x + \delta, \\ y_0 - \delta < y < y_0 + \delta, s_0 - \delta < s < s_0 + \delta \end{array} \right\}$ .

We can find  $\delta_1 = \frac{d}{K(\pi + \sigma)}$  which is smaller than  $\delta$  for which  $F_1(x, y, s), F_2(x, y, s), F_3(x, y, s), \frac{\partial F_1(x, y, s)}{\partial x},$

$\frac{\partial F_2(x, y, s)}{\partial y}$ , and  $\frac{\partial F_3(x, y, s)}{\partial z}$  are continuous and bounded on  $\Omega \in R$  where  $\Omega = \left\{ \begin{array}{l} (x(t), y(t), s(t)) \in \mathfrak{R}_+^3 : \\ x(t) + y(t) \leq K, s(t) \leq \frac{K(\sigma + \pi)}{d} \end{array} \right\}$ .

Hence, a  $Y = f(x, y, s)$  for the insurgency model exist and is unique on  $R = \left\{ \begin{array}{l} (x, y, s) = x_0 - \delta < x < x + \delta, \\ y_0 - \delta < y < y_0 + \delta, s_0 - \delta < s < s_0 + \delta \end{array} \right\}$ .

### 3. Positivity of Solutions

**Theorem 3.2:** Let the initial data set  $\{x(0), y(0), s(0)\}$  satisfy the initial conditions in (4). Then, the solution  $(x(t), y(t), s(t))$  with positive initial data set will remain positive for all time  $t > 0$ .

**Proof:**

Consider the first equation in the model system, i.e.,

$$\frac{dx(t)}{dt} = r_1 x(t) \left( 1 - \frac{x(t)}{K} \right) - \alpha x(t)s(t) - \phi x(t)y(t) - \gamma_1 x(t)s(t).$$

Then, by method of comparison, we have a first order logistic growth inequality  $\frac{dx(t)}{dt} \leq r_1 x(t) \left( 1 - \frac{x(t)}{K} \right)$ . By method of separation of variables, we have

$$\frac{1}{x(t) \left( 1 - \frac{x(t)}{K} \right)} dx(t) \leq r_1 dt \quad (7)$$

Resolving (7) into partial fraction, integrating and solving for  $x(t)$  we have;

$$x(t) = \frac{K}{\left[ 1 + \left( \frac{K - x(0)}{x(0)} \right) e^{-r_1 t} \right]} \quad (8)$$

Therefore, since  $x(0) > 0$ ,  $x(t) > 0 \quad \forall t > 0$ . It follows that  $x(t)$  will remain positive given that the initial data was positive. Similarly, in the same way, it can be shown that  $y(t), s(t) > 0 \quad \forall t > 0$ , given that  $y(0) > 0$  and  $s(0) > 0$  which conclude the proof.

## RESULTS

### Model Equilibrium Points

The model equilibria were obtained by setting the right-hands of the equations in (1)–(3) to zero. The equilibrium points as obtained using standard approaches to be:

i. The trivial equilibrium

$$P_0 = (x, y, s) = (0, 0, 0) \quad (9)$$

ii. The inactive-insurgents-free equilibrium

$$P_1 = (x, y, s) = \left( 0, \frac{d}{(\pi - \mu)}, \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right) \quad (10)$$

such that  $\pi > \mu$ .

iii. The active-insurgents-free equilibrium

$$P_2 = (x, y, s) = \left( \frac{d}{(\sigma - \varphi)}, 0, \frac{r_1 (K(\sigma - \varphi) - d)}{K(\sigma - \varphi)(\alpha + \gamma_1)} \right) \quad (11)$$

such that  $\sigma > \varphi$

iv. The security-forces-free equilibrium

$$P_3 = (x, y, s) = (K, K, 0) \quad (12)$$

v. The insurgency endemic equilibrium

$$P_4 = (x, y, s) = (x^*, y^*, s^*) \quad (13)$$

where;

$$x^* = \frac{d}{\varpi_1} - \frac{\varpi_2 \left[ \frac{K(r_2 \varpi_1 + d\phi)(r_1 K \varpi_1 - r_1 d - K \varpi_1(\alpha + \gamma_1))}{\Phi} \right]}{\varpi_1},$$

$$y^* = \frac{K(r_2 \varpi_1 + d\phi)(r_1 K \varpi_1 - r_1 d - K \varpi_1(\alpha + \gamma_1))}{\Phi},$$

$$s^* = \frac{K(r_2 \varpi_1 + d\phi)(\phi K \varpi_1 - r_1 \varpi_2)}{\Phi},$$

$$\text{with } \varpi_1 = (\sigma - \varphi), \varpi_2 = (\pi - \mu), \text{ and } \Phi = (r_2 \varpi_1 + \phi K \varpi_2)(r_1 K \varpi_1 - r_1 d - K \varpi_1(\alpha + \gamma_1)) + K \varpi_1(\beta + \gamma_2)(\phi K \varpi_1 - r_1 \varpi_2)$$

#### B. Local Stability of the Model Equilibria

The Jacobian matrix of the model is constructed as follows:

Define the following functions  $F_1, F_2$  and  $F_3$  by

$$F_1 = r_1 x(t) \left( 1 - \frac{x(t)}{K} \right) - \alpha x(t) s(t) - \phi x(t) y(t) - \gamma_1 x(t) s(t) \quad (19)$$

$$F_2 = r_2 y(t) \left( 1 - \frac{y(t)}{K} \right) + \phi x(t) y(t) - \beta y(t) s(t) - \gamma_2 y(t) s(t) \quad (20)$$

$$F_3 = \sigma x(t) s(t) + \pi y(t) s(t) - \varphi x(t) s(t) - \mu y(t) s(t) - ds(t) \quad (21)$$

The Jacobian matrix of the model system is given as

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial s} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial s} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial s} \end{bmatrix} \quad (22)$$

Substituting the partial derivatives of the arbitrary functions  $F_1, F_2$  and  $F_3$  into (22), we obtained:

$$\begin{bmatrix} r_1 - \frac{2r_1x}{K} - \phi y - (\alpha + \gamma_1)s \\ \phi y \\ (\sigma - \phi)s \\ -\phi x \\ r_2 - \frac{2r_2y}{K} + \phi x - (\beta + \gamma_2)s \\ (\pi - \mu)s \\ -(\alpha + \gamma_1)x \\ -(\beta + \gamma_2)y \\ (\sigma - \phi)x + (\pi - \mu)y - d \end{bmatrix} \quad (23)$$

To analyze the stability of the trivial equilibrium point  $P_0$ , we evaluated the Jacobian matrix  $J$  at  $P_0$ , i.e.

$$J|_{P_0} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & -d \end{bmatrix} \quad (24)$$

Therefore, the characteristic equation of (24) is obtained as follows:

$$|J|_{P_0} - \lambda I = \begin{vmatrix} r_1 - \lambda & 0 & 0 \\ 0 & r_2 - \lambda & 0 \\ 0 & 0 & -d - \lambda \end{vmatrix} = 0 \quad (25)$$

$$\Rightarrow (r_1 - \lambda)(r_2 - \lambda)(-d - \lambda) = 0 \quad (26)$$

$\therefore \lambda_1 = +r_1, \lambda_2 = +r_2, \lambda_3 = -d$ , which is a saddle point. Therefore, the stability of  $P_0$  cannot be concluded for now.

Next, we analyze the stability of the inactive-insurgents-free equilibrium  $P_1$  by evaluating  $J$  at  $P_1$ , i.e.

$$\begin{bmatrix}
r_1 - \frac{\phi d}{(\pi - \mu)} - \frac{r_1(K(\sigma - \varphi) - d)}{K(\sigma - \varphi)} & & & \\
\frac{\phi d}{(\pi - \mu)} & & & \\
\frac{r_1(K(\sigma - \varphi) - d)}{(\alpha + \gamma_1)} & & & \\
0 & & & \\
r_2 - \frac{2r_2 d}{K(\pi - \mu)} - \frac{r_1(K(\sigma - \varphi) - d)(\beta + \gamma_2)}{K(\sigma - \varphi)(\alpha + \gamma_1)} & & & \\
\frac{r_1(K(\sigma - \varphi) - d)(\pi - \mu)}{K(\sigma - \varphi)(\alpha + \gamma_1)} & & & \\
0 & & & \\
-\frac{d(\beta + \gamma_2)}{(\pi - \mu)} & & & \\
0 & & & 
\end{bmatrix} \quad (27)$$

The characteristic equation  $|J_{|P_1} - \lambda I| = 0$  gives the following eigenvalues after simplifications:

$$\left\{ \begin{aligned}
\lambda_1 &= \frac{r_1 K(\sigma - \varphi)(\pi - \mu)}{K(\sigma - \varphi)(\pi - \mu)} \\
&\quad - \frac{[\phi d K(\sigma - \varphi) + r_1(K(\sigma - \varphi) - d)(\pi - \mu)]}{K(\sigma - \varphi)(\pi - \mu)} \\
\lambda_2 &= \frac{r_1(\pi - \mu)}{(\alpha + \gamma_1)} - \frac{r_1 d(\pi - \mu)}{K(\sigma - \varphi)(\alpha + \gamma_1)} \\
\lambda_3 &= -\frac{d(\beta + \gamma_2)}{(\pi - \mu)}
\end{aligned} \right.$$

For the stability of the equilibrium point  $P_1$ , we require that  $\lambda_{1,2,3} < 0$ . It is obvious that  $\lambda_3 = -\frac{d(\beta + \gamma_2)}{(\pi - \mu)} < 0$ , with  $\pi > \mu$ . For

$\lambda_1, \lambda_2 < 0$ , we require that  $\frac{r_1(\pi - \mu)}{K(\sigma - \varphi)} < \phi$ ,  $(\sigma - \varphi) < \frac{d}{K}$  and  $\sigma > \varphi$ . Therefore, the inactive-insurgents-free equilibrium  $P_1$  is

locally asymptotically stable with the above stability conditions.

Next, we analyze the stability of the active-insurgents-free equilibrium  $P_2$  by evaluating  $J$  at  $P_2$ , i.e

$$(29)$$

The characteristic equation  $|J_{|P_2} - \lambda I| = 0$  gives the following eigenvalues after reducing the matrix in (29) to an upper triangular matrix using elementary row-operations as used by (Usman & Adamu, 2017; Usman, Adamu & Aliyu, 2017; Usman, Adamu & Umar, 2017).

$$\begin{cases} \lambda_1 = \frac{r_1 d}{K} \left( \frac{1}{(\sigma - \phi)} - \frac{2}{(\pi - \mu)} \right) \\ \lambda_2 = r_2 + \frac{\phi d}{(\pi - \mu)} + \frac{d(\beta + \gamma_2)}{K(\sigma - \phi)(\alpha + \gamma_1)} - \frac{r_1(\beta + \gamma_2)}{(\alpha + \gamma_1)} \\ \lambda_3 = \frac{(\sigma - \phi)d}{(\pi - \mu)} - d = -d \left( 1 - \frac{(\sigma - \phi)}{(\pi - \mu)} \right) \end{cases} \quad (30)$$

From (30), it is obvious that  $\lambda_{1,3} < 0$  if  $\frac{(\sigma - \phi)}{(\pi - \mu)} < 1$ . The ratio  $\frac{(\sigma - \phi)}{(\pi - \mu)}$  represents the effort of the presence of security forces in

response to or against the presence of the insurgents, and that  $\lambda_2 < 0$  if  $\frac{K(\sigma - \phi)(\pi - \mu)[r_1(\beta + \gamma_2) - r_2(\alpha + \gamma_1)]}{> d[K\phi(\sigma - \phi)(\alpha + \gamma_1) + (\beta + \gamma_2)(\pi - \mu)]}$ .

Therefore, the active-insurgents-free equilibrium  $P_2$  is locally asymptotically stable with these conditions.

Next, we analyze the stability of the security-forces-free equilibrium  $P_3$  by evaluating  $J$  at  $P_3$ , i.e.

$$J|_{P_3} = \begin{bmatrix} -(r_1 + \phi K) & -\phi K & -(\alpha + \gamma_1) \\ \phi K & -(r_2 + \phi K) & -(\beta + \gamma_2) \\ 0 & 0 & (\sigma - \phi)K + (\pi - \mu)K - d \end{bmatrix} \quad (31)$$

Simplifying the characteristic equation  $|J|_{P_3} - \lambda I| = 0$ , gives the following cubic equation:

$$\begin{aligned} & [(\sigma - \phi)K + (\pi - \mu)K - d - \lambda] \times \\ & [r_1 r_2 + (r_2 - r_1)\phi K + (r_1 + r_2)\lambda + \lambda^2] = 0 \end{aligned} \quad (32)$$

Therefore, the cubic equation in (32) gives the following eigenvalues:

$$\begin{cases} \lambda_1 = ((\sigma - \phi)K + (\pi - \mu)K) - d \\ \lambda_{2,3} = \frac{-(r_1 + r_2) \pm \sqrt{(r_1 + r_2)^2 - 4(r_1 + r_2 + (r_2 - r_1)\phi K)}}{2} \end{cases}$$

For  $\lambda_1 < 0$ , we need  $\sigma < \phi$ ,  $\pi < \mu$  and  $d > 0$  which is contrary to our previous assumption. Therefore, since  $\sigma > \phi$  and  $\pi > \mu$ , then  $((\sigma - \phi)K + (\pi - \mu)K) > d$ , which implies  $\lambda_1 > 0$ . Hence,  $P_3$  is an unstable equilibrium regardless of whether  $\lambda_2, \lambda_3 < 0$ .

### C. Global Stability of the Model Equilibria

Theorem 1: The trivial equilibrium point  $P_0 = (x, y, s) = (0, 0, 0)$  is globally asymptotically stable with respect to the set  $\Omega$

- If there is  $V(x)$  such that  $V(0) = 0$  and  $V(x) > 0, \forall x \in \Omega$  with  $x \neq 0$  and  $\dot{V}(x) \leq 0, \forall x \in \Omega$  then the origin is a stable
- Moreover, if  $\dot{V}(x) < 0, \forall x \in \Omega$  with  $x \neq 0$ , then the origin is asymptotically stable
- Furthermore, if  $V(x) > 0, \forall x \neq 0, \|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty \forall x \neq 0$ , then the origin is globally asymptotically stable

**Proof:**

Let  $V(X)$  be the Lyapunov candidate function, where; first, we drive the gradient function for the Lyapunov candidate function;

$$V(XV(X) = V(x, y, s) \text{ given as } \nabla V(X) = \begin{bmatrix} \nabla V_1(x) \\ \nabla V_2(y) \\ \nabla V_3(s) \end{bmatrix},$$

where  $\nabla V_i(X) = \sum_{j=1}^3 h_{ij} x_j$ ,

$i = 1, 2, 3$ , and  $x_1 = x, x_2 = y$  &  $x_3 = s$ . Hence;

$$\begin{aligned}\nabla V_1(X) &= h_{11}x + h_{12}y + h_{13}s \\ \nabla V_2(X) &= h_{21}x + h_{22}y + h_{23}s \\ \nabla V_3(X) &= h_{31}x + h_{32}y + h_{33}s\end{aligned}$$

$$\text{Therefore } \nabla V(X) = \begin{bmatrix} h_{11}x + h_{12}y + h_{13}s \\ h_{21}x + h_{22}y + h_{23}s \\ h_{31}x + h_{32}y + h_{33}s \end{bmatrix}$$

The curl condition gives;

- i)  $\frac{\partial h_{ik}}{\partial x_j} = 0$  for  $i \neq j, k = 1, 2, 3$
- ii)  $h_{12} = h_{21} = h_{13} = h_{31} = h_{23} = h_{32} = 0$
- iii)  $h_{11} = h_{22} = h_{33} = \text{constant}.$

$$\nabla V(X) \text{ now becomes } \nabla V(X) = \begin{bmatrix} h_{11}x \\ h_{22}y \\ h_{33}s \end{bmatrix},$$

which is the required gradient function.

The derivative  $\dot{V}(X) = [\nabla V(X)]^T \cdot F$ .

Thus;  $\dot{V}(X) = [h_{11}x \quad h_{22}y \quad h_{33}s] \cdot F$

$$\begin{aligned}&= [h_{11}x \quad h_{22}y \quad h_{33}s] \cdot \\&\begin{bmatrix} r_1x(t) \left(1 - \frac{x(t)}{K}\right) - \alpha x(t)s(t) - \phi x(t)y(t) - \gamma_1x(t)s(t) \\ r_2y(t) \left(1 - \frac{y(t)}{K}\right) + \phi x(t)y(t) - \beta y(t)s(t) - \gamma_2y(t)s(t) \\ \sigma x(t)s(t) + \pi y(t)s(t) - \varphi x(t)s(t) - \mu y(t)s(t) - ds(t) \end{bmatrix} \\&\Rightarrow \dot{V}(X) = h_{11}x(t) \left[ r_1x(t) \left(1 - \frac{x(t)}{K}\right) - \alpha x(t)s(t) - \right. \\&\quad \left. \phi x(t)y(t) - \gamma_1x(t)s(t) \right] \\&\quad + h_{22}y(t) \left[ r_2y(t) \left(1 - \frac{y(t)}{K}\right) + \right. \\&\quad \left. \phi x(t)y(t) - \beta y(t)s(t) - \gamma_2s(t)y(t) \right] \\&\quad + h_{33}s(t) \left[ \sigma x(t)s(t) + \pi y(t)s(t) - \right. \\&\quad \left. \varphi x(t)s(t) - \mu y(t)s(t) - ds(t) \right]\end{aligned}$$

Since  $h_{11}$ ,  $h_{22}$  and  $h_{33}$  are constants not equals to 0. Let  $h_{11} = h_{22} = h_{33} = 1$ . Opening the brackets and simplifying we have;

$$\begin{aligned}\dot{V}(X) &= -\frac{1}{K}(x^3 - x^2)r_1 - (as + \phi y)x - \gamma_1xs - \\&\quad \frac{1}{K}((y^3 - y^2)r_2 - \phi yx) - ys(\beta + \gamma_2) - \\&\quad (\psi - \sigma)xs^2 - (\mu - \pi)ys - ds < 0, \\&\quad \text{for } \psi > \sigma \text{ and } \mu > \pi\end{aligned}$$

Clearly  $\dot{V}(X) < 0$  for  $\psi > \sigma$  and  $\mu > \pi$ .

### The Lyapunov function V(X)

The Lyapunov function V(X) can be obtained from the integral

$$\begin{aligned}V(X) &= \int_0^x [\nabla V_1(X), \nabla V_2(X), \nabla V_3(X)] \begin{bmatrix} dx(t) \\ dy(t) \\ ds(t) \end{bmatrix} = \\&\int_0^x (y=s=0) \nabla V_1(X) + \int_0^y (x=s=0) \nabla V_2(X) + \int_0^s (y=x=0) \nabla V_3(X) \\&= \int_0^x \nabla V_1(s_1, 0, 0) dx \\&\quad + \int_0^y \nabla V_2(0, s_2, 0) dy + \int_0^s \nabla V_3(0, 0, s_3) ds\end{aligned}$$

Now, using equation (4), we have;

$$\int_0^x h_{11}x \, dx + \int_0^y h_{22}y \, dy + \int_0^s h_{33}s \, ds =$$

$$h_{11} \frac{x^2}{2} + h_{22} \frac{y^2}{2} + h_{33} \frac{s^2}{2}$$

$$V(X) = \frac{1}{2} (h_{11}x^2 + h_{22}y^2 + h_{33}s^2)$$

We now Check  $V(X)$  and  $\dot{V}(X)$  for positive and negative definiteness respectively as follows;  
with  $h_{11} = h_{33} = h_{33} = 1$ , we obtained;

$$V(X) = \frac{1}{2} (x^2 + y^2 + s^2) > 0 \dots \dots (7)$$

From (5) and (7),  $\dot{V}(X) < 0$  and  $V(X) > 0$  respectively, therefore  $V(X)$  is positive definite and  $\dot{V}(X)$  is negative definite. This means the equilibrium point is asymptotically stable. Further;  $V(X) > 0, \forall X \neq 0$ , and  $\|X\| \rightarrow \infty \Rightarrow V(X) \rightarrow \infty \forall X \neq 0$ . Hence by the Lyapunov global stability theorem. By definition 5 the trivial equilibrium point is globally asymptotically stable

We conclude that the equilibrium point (then the origin)  $X = (0, 0, 0)$  is globally asymptotically stable for  $\psi > \sigma$  and  $\mu > \pi$ .

**For**

**the**

**equilibrium**

**point**

$$P_1 = \left( 0, \frac{d}{(\pi - \mu)}, \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right)$$

And letting  $h_{11} = h_{33} = h_{33} = 1$ , we have for;

$$V(P_1) = V \left( 0, \frac{d}{(\pi - \mu)}, \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right) =$$

$$\frac{1}{2} \left[ \left( \frac{d}{(\pi - \mu)} \right)^2 + \left( \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right)^2 \right] > 0$$

$$\dot{V} \left( 0, \frac{d}{(\pi - \mu)}, \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right)$$

$$= 0 + \frac{d}{(\pi - \mu)} \left[ \frac{r_2 \frac{d}{(\pi - \mu)} \left( 1 - \frac{d}{K(\pi - \mu)} \right) - \frac{d}{\beta \frac{d}{(\pi - \mu)} \left( \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right) - \frac{r_2 (K(\pi - \mu) - d)}{\gamma_2 K(\pi - \mu)(\beta + \gamma_2)} \frac{d}{(\pi - \mu)} \right]$$

$$+ \left( \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right) \left[ -\mu \frac{d}{(\pi - \mu)} \left( \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right) - d \left( \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right) \right]$$

$$= -\frac{d}{(\pi - \mu)} \left[ r_2 \frac{d}{(\pi - \mu)} \left( \frac{d}{K(\pi - \mu)} - 1 \right) \right] -$$

$$\frac{d}{(\pi - \mu)} \left[ \frac{\beta d}{(\pi - \mu)} \left( \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right) + \frac{d}{\gamma_2 \frac{d}{(\pi - \mu)} \left( \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right)} \right] -$$

$$\left( \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right) \left[ \mu \frac{d}{(\pi - \mu)} \left( \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right) - \right]$$

$$\left[ \pi \frac{d}{(\pi - \mu)} \left( \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right) + d \left( \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right) \right] < 0$$

Clearly,  $V(X) > 0, \forall X \neq 0$ , and  $\|X\| \rightarrow \infty \Rightarrow V(X) \rightarrow \infty \forall X \neq 0$ .

Therefore,

**the**

**equilibrium**

**point**

$P_1 = \left( 0, \frac{d}{(\pi - \mu)}, \frac{r_2 (K(\pi - \mu) - d)}{K(\pi - \mu)(\beta + \gamma_2)} \right)$  is globally asymptotically stable.

**For the equilibrium point**

$$P_2 = \left( \frac{d}{(\sigma - \varphi)}, 0, \frac{r_1 (K(\sigma - \varphi) - d)}{K(\sigma - \varphi)(\alpha + \gamma_1)} \right)$$

We have  $V(X) = \frac{1}{2}(h_{11}x^2 + h_{22}y^2 + h_{33}s^2)$

$$= \frac{1}{2} \left[ \left( \frac{d}{(\sigma - \varphi)} \right)^2 + \left( \frac{r_1 (K(\sigma - \varphi) - d)}{K(\sigma - \varphi)(\alpha + \gamma_1)} \right)^2 \right] > 0 \quad \forall x, y, s$$

We

also

have

for

$\dot{V}(X)$

$$\begin{aligned} \dot{V}(X) = & \frac{d}{(\sigma - \varphi)} \left[ a \frac{r_1 (K(\sigma - \varphi) - d)}{(\sigma - \varphi) K(\sigma - \varphi)(\alpha + \gamma_1)} - \right. \\ & \left. \gamma_1 \frac{d}{(\sigma - \varphi) K(\sigma - \varphi)(\alpha + \gamma_1)} \right] + \\ & \frac{r_1 (K(\sigma - \varphi) - d)}{K(\sigma - \varphi)(\alpha + \gamma_1)} \left[ \psi \frac{d}{(\sigma - \varphi) K(\sigma - \varphi)(\alpha + \gamma_1)} - \right. \\ & \left. d \frac{r_1 (K(\sigma - \varphi) - d)}{K(\sigma - \varphi)(\alpha + \gamma_1)} \right] \\ & - \frac{d}{(\sigma - \varphi)} \left[ a \frac{r_1 (K(\sigma - \varphi) - d)}{(\sigma - \varphi) K(\sigma - \varphi)(\alpha + \gamma_1)} - \right. \\ & \left. \gamma_1 \frac{d}{(\sigma - \varphi) K(\sigma - \varphi)(\alpha + \gamma_1)} \right] \\ & - \frac{r_1 (K(\sigma - \varphi) - d)}{K(\sigma - \varphi)(\alpha + \gamma_1)} \left[ \psi \frac{d}{(\sigma - \varphi) K(\sigma - \varphi)(\alpha + \gamma_1)} + \right. \\ & \left. d \frac{r_1 (K(\sigma - \varphi) - d)}{K(\sigma - \varphi)(\alpha + \gamma_1)} - \right. \\ & \left. \sigma \frac{r_1 (K(\sigma - \varphi) - d)}{K(\sigma - \varphi)(\alpha + \gamma_1)} \right] \\ & < 0 \text{ for } \sigma > \varphi \end{aligned}$$

Clearly  $V(X) > 0, \forall X \neq 0$ , and  $\|X\| \rightarrow \infty \Rightarrow V(X) \rightarrow \infty \forall X \neq 0$ .

Therefore,

the

equilibrium

point

$$P_2 = \left( \frac{d}{(\sigma - \varphi)}, 0, \frac{r_1 (K(\sigma - \varphi) - d)}{K(\sigma - \varphi)(\alpha + \gamma_1)} \right)$$

is globally asymptotically stable.

**For the equilibrium point**

$$P_3 = (x, y, s) = (K, K, 0)$$

We have  $V(X) = \frac{1}{2}(K^2 + K^2) > 0 \quad \forall x, y, s$

We also have for  $\dot{V}(X)$

$$\begin{aligned} \dot{V}(X) = & K \left[ r_1 K \left( 1 - \frac{K}{K} \right) - ax(t)s(t) - \phi K^2 - \gamma_1 x(t)s(t) \right] \\ & + K \left[ r_2 K \left( 1 - \frac{K}{K} \right) + \phi K^2 - \beta K^2 - \gamma_2 s(t)y(t) \right] \\ & + h_{33}s(t) \left[ -\psi x(t)s(t) - \mu y(t)s(t) - ds(t) \right] \\ & \dot{V}(X) = -\beta K^3 < 0 \quad \forall x, y, s \end{aligned}$$

Clearly  $V(X) > 0, \forall X \neq 0$ , and  $\|X\| \rightarrow \infty \Rightarrow V(X) \rightarrow \infty \forall X \neq 0$ .

Therefore, the equilibrium point  $P_2 = (K, K, 0)$

is globally asymptotically stable.

**For the endemic equilibrium**  $P_4 = (x, y, s) = (x^*, y^*, s^*)$

where;

$$x^* = \frac{d}{\omega_1} -$$

$$y^* = \frac{\frac{\omega_2}{\omega_1} \left[ \frac{K(r_2\omega_1 + d\phi)(r_1K\omega_1 - r_1d - K\omega_1(\alpha + \gamma_1))}{(r_2\omega_1 + \phi K\omega_2)(r_1K\omega_1 - r_1d - K\omega_1(\alpha + \gamma_1)) + K\omega_1(\beta + \gamma_2)(\phi K\omega_1 - r_1\omega_2)} \right]}{K(r_2\omega_1 + d\phi)(r_1K\omega_1 - r_1d - K\omega_1(\alpha + \gamma_1)) + K\omega_1(\beta + \gamma_2)(\phi K\omega_1 - r_1\omega_2)}$$

$$s^* = \frac{K(r_2\omega_1 + d\phi)(\phi K\omega_1 - r_1\omega_2)}{(r_2\omega_1 + \phi K\omega_2)(r_1K\omega_1 - r_1d - K\omega_1(\alpha + \gamma_1)) + K\omega_1(\beta + \gamma_2)(\phi K\omega_1 - r_1\omega_2)}$$

Checking for positive definiteness for  $V(X)$  and  $\dot{V}(X)$  we have:  $V(X) = \frac{1}{2} \frac{\omega_2}{\omega_1} \left[ \frac{K(r_2\omega_1 + d\phi)(r_1K\omega_1 - r_1d - K\omega_1(\alpha + \gamma_1))}{(r_2\omega_1 + \phi K\omega_2)(r_1K\omega_1 - r_1d - K\omega_1(\alpha + \gamma_1)) + K\omega_1(\beta + \gamma_2)(\phi K\omega_1 - r_1\omega_2)} \right]^2 + \left[ \frac{K(r_2\omega_1 + d\phi)(r_1K\omega_1 - r_1d - K\omega_1(\alpha + \gamma_1))}{(r_2\omega_1 + \phi K\omega_2)(r_1K\omega_1 - r_1d - K\omega_1(\alpha + \gamma_1)) + K\omega_1(\beta + \gamma_2)(\phi K\omega_1 - r_1\omega_2)} \right]^2$

$> 0 \forall x, y, s$

We also have for  $\dot{V}(X)$

$$\begin{aligned} \dot{V}(X) = & x^* \left[ r_1 x^* \left( 1 - \frac{x^*}{K} \right) - ax^*s^* - \phi x^*y^* - \gamma_1 x^*s^* \right] \\ & + y^* \left[ r_2 y^* \left( 1 - \frac{y^*}{K} \right) + \phi x^*y^* - \beta y^*s^* - \gamma_2 s^*y^* \right] \\ & + s^* [\sigma s^* + \pi y^*s^* - \psi x^*s^* - \mu y^*s^* - ds^*] \end{aligned}$$

### Numerical Simulations

In this section, numerical simulations for the model were carried out using hypothetical parameter values in Tables; C1, C2 and C3 in Appendix C for the purpose of illustrations. We used MATLAB R2019a encoded with ODE45 solver to simulate the system using the parameters and an initial population for the inactive insurgents, active insurgents and the security forces, where we resolved them into proportions. The results of the numerical simulations are presented in Appendix A and B. In Appendix A, the graphs were obtained by varying only the rehabilitation rates (parameters)  $\alpha$  and  $\beta$ . In Appendix B, some of the other parameters, apart from  $\alpha$  and  $\beta$  are varied to generate the graphs using the parameter values in Table C2, Appendix C.

### DISCUSSIONS

In developing the model in this paper, the interaction between the security-forces and insurgents was assumed to be a prey-predator like interaction. Ideally, the security-forces were supposed to be the predators while the insurgents are the prey. But, since it was assumed that, the insurgents have logistic capacity to attack the security-forces, at some point in time, the security-forces could become the prey while the insurgents become the predator. In the model analysis, the populations of insurgents (*the inactive and the active*) grow logistically up to the carrying capacity of the habitat in the absence of the security-forces and completion for resources. This is demographically true for the dynamics of the populations in the absence of diseases and competition for resources. We carried out two types of experiments, viz:

Experiment A: Varying (incrementally) the rehabilitation rate ( $\alpha$ ) for inactive insurgents as  $\alpha=0.7, \alpha=0.8, \alpha=0.9$  and

$\alpha=1.0, \alpha=1.1$ , and fixing all other parameter values at the baseline figures to generate four simulation results as shown Fig A1 to A5.

Experiment B: Varying (incrementally) the rehabilitation rate ( $\beta$ ) for active insurgents as  $\beta=0.6, \beta=0.7, \beta=0.8$  and  $\beta=0.9, \beta=1.0$ , and fixing all other parameter values at the baseline figures to generate four simulation results as shown Fig AA1 to AA5.

Experiment C: Varying (incrementally) the rehabilitation rates for the inactive and active insurgents  $\alpha, \beta$  respectively; the killing rates for the inactive and active insurgents  $\gamma_1, \gamma_2$  respectively; the recruitment rates for the inactive and active insurgents,  $\sigma$  and  $\pi$  respectively, and reduced the values of  $\phi, \mu, r_1, r_2$  as indicated in Columns C2 to C5 of Table C2. On the basis of this, we generated Fig. B2 to Fig. B5.

### Experiment A:

The figures in Fig A1 to A5 showed that; with steady increase in the rehabilitation rate of the inactive insurgents ( $\alpha$ ), and keeping other parameter rates constant at the baseline values, there is no significant effect on the dynamics of the system. Meaning concentrating on rehabilitation of the inactive insurgents may not yield a significant effect on the dynamics of insurgency, and hence we should investigate effect of varying the parameters relating to the active insurgents and security forces.

### Experiment B:

The figures in Fig AA1 to AA5 showed that; with steady increase in the rehabilitation rate of the active insurgents ( $\beta$ ), while keeping other parameter rates constant at the

baseline values, the population of active insurgents depleted to near elimination, while the population of inactive insurgents is kept under control with minimal effort from the security forces. This suggests that; rehabilitating active insurgents when captured reduced tremendously the population of active insurgents.

### Experiment C

The figures in Fig B2 to B5 showed that; with steady increase in the rehabilitation rates for the inactive and active insurgents  $\alpha$ ,  $\beta$  respectively; the killing rates for the inactive and active insurgents  $\gamma_1$ ,  $\gamma_2$  respectively; the recruitment rates for the security forces due to actions of the inactive and active insurgents,  $\sigma$  and  $\pi$  respectively, as indicated in columns C2 to C5 of **table C2** (highlighted in light green), and gradual reduction of the values of; the constant killing rates of the security forces by the active and inactive insurgent's when they engage them or through suicide missions  $\varphi$ ,  $\mu$ , respectively, Logistic growth rate for the inactive and active insurgents  $r_1$ ,  $r_2$  respectively as indicated in columns C2 to C5 of **table C2** (highlighted in light red), while keeping other parameter rates constant at the baseline values, the population of active insurgents depleted to near elimination, while the population of inactive insurgents is kept decreasing gradually (*but slower than that of inactive insurgents*) with time to near elimination. under control with minimal effort from the security forces. This suggests that; rehabilitating active insurgents when captured reduced tremendously the population of active insurgents.

### FIG1:

This is the base population dynamics graph at the set parameter values in column *Figure B1* of **table C2**. This graph is to be used for comparison with subsequent dynamical graphs for varying the parameter values away from the base parameter values. In this figure, we noticed that the population of active insurgents outweighs the population of inactive insurgents, while the population of the security forces overshadows the population of the insurgents. This means the security forces population is in control of the insurgents.

### FIG2:

In column *Figure B2* of **table C2** we increased the logistic growth and rehabilitation rates for inactive insurgents, while reducing logistic growth, initiation rates for active insurgents, and keeping constant; the rehabilitation rate for active insurgents, while reducing the recruitment rates for security forces due to the activities of insurgents (*inactive and active*), and increasing the killing rates of the security forces by the

insurgents (*inactive and active*), and increasing the decay (withdrawal) rate of the security forces due to the absence of insurgents, we noticed that the active insurgents population reduces to a level lower than the population of the inactive insurgents, while the security forces population is on top of the populations of the insurgents. This scenario shows that the security forces are in control of the situation

### FIG3:

From the data *column Figure 3 table C2*, we increased (*based on column "Figure 2"*) the logistic growth and rehabilitation rates for inactive insurgents, and reduced further; the logistic growth, initiation and the rehabilitation rates for active insurgents, and maintaining (*keeping constant*); the killing rate for the insurgents, recruitment and killing rates of security forces due to the activities of active insurgents, and reducing the killing rate of the security forces by the inactive insurgents and maintaining the decay (*withdrawal*) rate of the security forces due to the absence of insurgents. We noticed that the active insurgent's population overshadows the population of the security forces which also over shadows the population of the inactive insurgents. In this scenario, it means the active insurgents are in control, a situation not required

### FIG4:

From the data used in *Figure 3*, we reduced further the logistic growth for the insurgents (*inactive and active*), rehabilitation (*inactive and active*) and initiation rates for inactive insurgents, and maintaining (*keeping constant*); the killing rate for the insurgents, recruitment and killing rates of security forces due to the activities of active insurgents, and increasing the killing rate of the security forces by the inactive insurgents and maintaining the decay (*withdrawal*) rate of the security forces due to the absence of insurgents. We noticed that the population of; the inactive insurgents and the security forces are competing, but with time the population of the security forces overshadows the population of the inactive population. Under the same condition, the population of the active insurgents drops drastically to a negligible level (near zero). In this scenario, it means the security forces have tamed the active insurgents to extinction and kept the inactive insurgents under control, a situation highly required.

We also saw that with Security-forces recruitment (into the affected areas by insurgency) at a rate dependent on the activities of the inactive and active insurgents, the population of the insurgents was kept under check. In the model, security-forces are withdrawn at a constant rate

only when the activities of the insurgents are negligible, and the withdrawal constant rate is the only bound on the security-forces population.

#### CONCLUSION

From the discussion and the graphs of tables A2-A3 and B4 above, effort should be geared towards

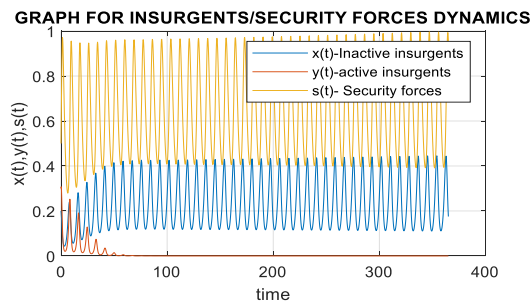
- ✓ Capturing and rehabilitating the insurgents
- ✓ Where capturing and rehabilitating is not feasible, the insurgents should be killed.

Security-forces should be recruited into the areas affected by insurgency at a rate dependent on the activities of the inactive and active insurgents, and withdrawn only when the activities of the insurgents have tamed by way reducing their population to a quarter of the population of the security forces.

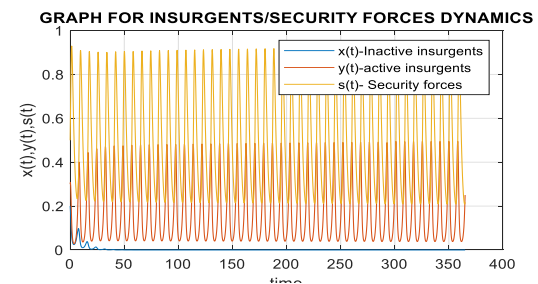
#### APPENDICES

##### APPENDIX A

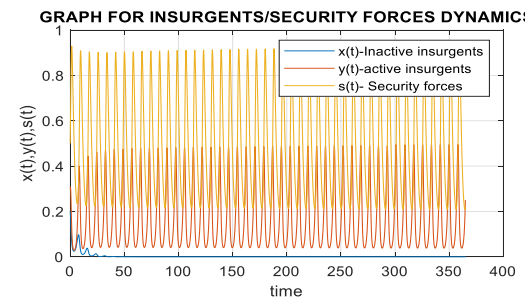
SIMULATION RESULTS FOR THE DYNAMICS UNDER THE INFLUENCE OF VARYING RECRUITMENT RATE  $\alpha$  FOR INACTIVE INSURGENTS RESPECTIVELY AS CONTAINED IN TABLE C1



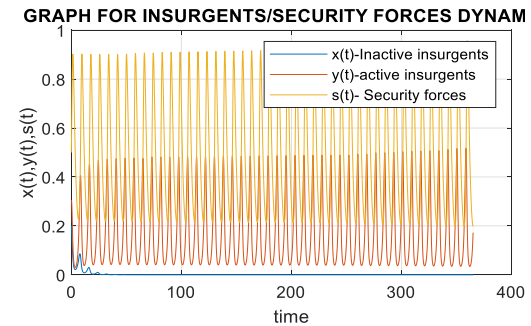
**Figure A1:** Simulation Result for the model with rehabilitation rates  $\alpha=0.7$  and  $\beta=0.8$



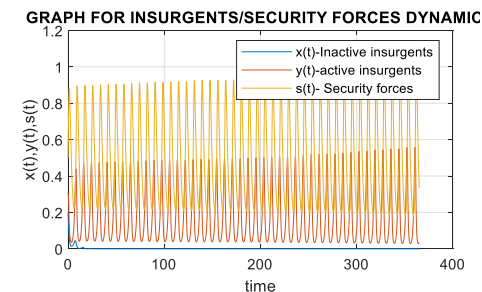
**Figure A2:** Simulation Result for the model with rehabilitation rates at  $\alpha=0.8$  and  $\beta=0.8$



**Figure A3:** Simulation Result for the model with rehabilitation rates at  $\alpha=1.0$  and  $\beta=0.8$



**Figure A4:** Simulation Result for the model with rehabilitation rates at  $\alpha=1.2$  and  $\beta=0.8$



**Figure A5:** Simulation Result for the model with rehabilitation rates at  $\alpha=1.4$  and  $\beta=0.8$ .

##### APPENDIX B

SIMULATION RESULTS UNDER THE INFLUENCE INCREASED VALUES OF  $\alpha$ ,  $\beta$ ,  $\gamma_1$ ,  $\gamma_2$ ,  $\sigma$  AND  $\pi$ , AND

REDUCED VALUES OF  $\varphi, \mu, r_1, r_2$  AS CONTAINED IN TABLE C2

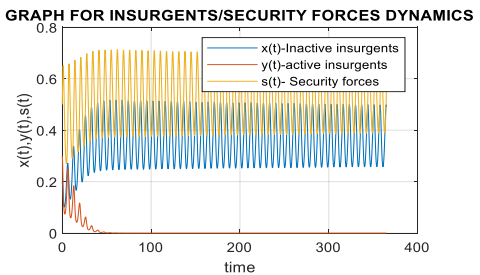


Figure B1: Simulation Result for the model with increased values of  $\alpha, \beta, \gamma_1, \gamma_2, \sigma$  and  $\pi$ , and reduced values of  $\varphi, \mu, r_1, r_2$  as contained in column B2 of table C2

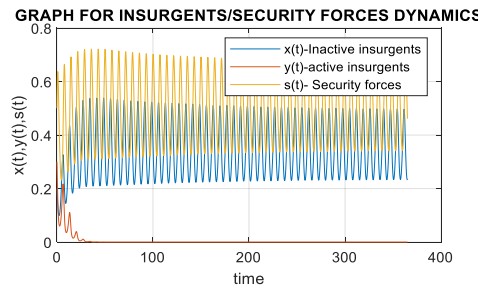


Figure B2: Simulation Result for the model with further increased values of  $\alpha, \beta, \gamma_1, \gamma_2, \sigma$  and  $\pi$ , and further reduced values of  $\varphi, \mu, r_1, r_2$  as contained in column B3 of table C2

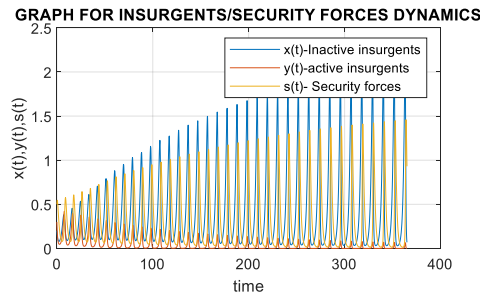


Figure B3: Simulation Result for the model with further increased values of  $\alpha, \beta, \gamma_1, \gamma_2, \sigma$  and  $\pi$  and, further

reduced values of  $\varphi, \mu, r_1, r_2$  away from B3 as contained in column B4 of Table C2

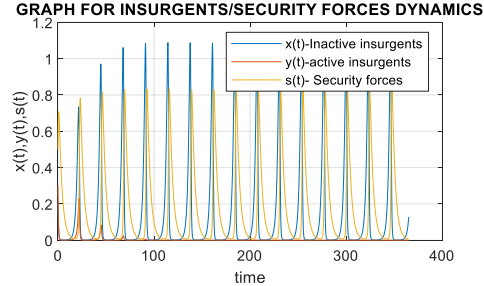


Figure B4: Simulation Result for the model with further increased values of  $\alpha, \beta, \gamma_1, \gamma_2, \sigma$  and  $\pi$  and further reduced values of  $\varphi, \mu, r_1, r_2$  away from B4 as contained in column B5 of Table C2.

APPENDIX C  
SIMULATION RESULTS FOR THE DYDNAMICS UNDER THE INFLUENCE OF VARYING REHABILITATION RATE  $\beta$  FOR ACTIVE INSURGENTS RESPECTIVELY AS CONTAINED IN TABLE C3.

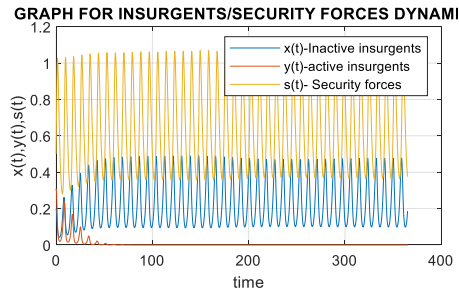


Figure AA1: Simulation Result for the model with rehabilitation rates  $\alpha=0.7$  and  $\beta=0.6$

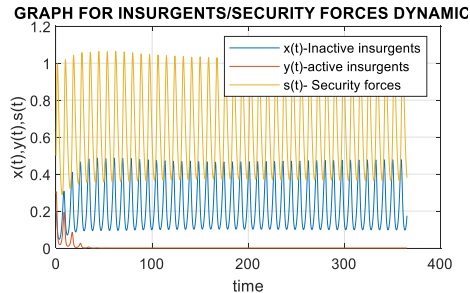


Figure AA2: Simulation Result for the model with rehabilitation rates  $\alpha=0.7$  and  $\beta=0.7$

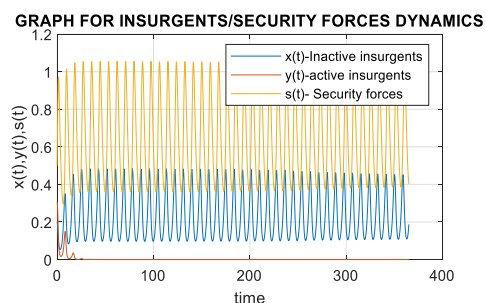


Figure AA3: Simulation Result for the model with rehabilitation rates  $\alpha=0.7$  and  $\beta=0.8$ .

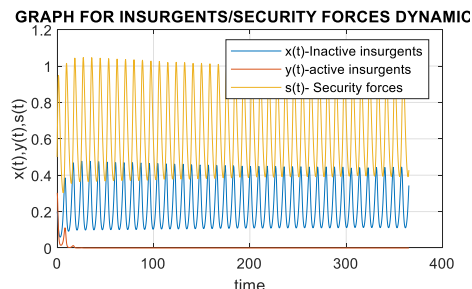


Figure AA4: Simulation Result for the model with rehabilitation rates  $\alpha=0.7$  and  $\beta=0.9$ .

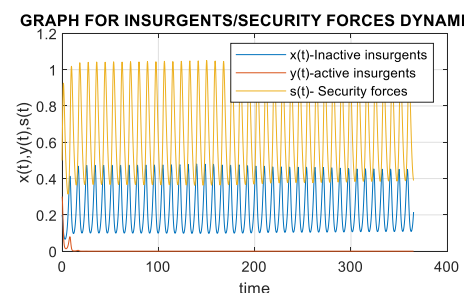


Figure AA1: Simulation Result for the model with rehabilitation rates  $\alpha=0.7$  and  $\beta=1.0$ .

#### APPENDIX D

SIMULATION RESULTS FOR THE DYNAMICS UNDER THE INFLUENCE OF NEGATIVE VARIATION in  $\phi, \mu, r_1, r_2$  and  $\phi$  AS CONTAINED IN TABLE C4.

Simulation results with –ve variations in  $\phi, \mu, r_1, r_2$  and  $\phi$  of the baseline parameter values used in numerical simulations

#### APPENDIX E

TABLE OF PARAMETER VALUES USED IN THE SIMULATION

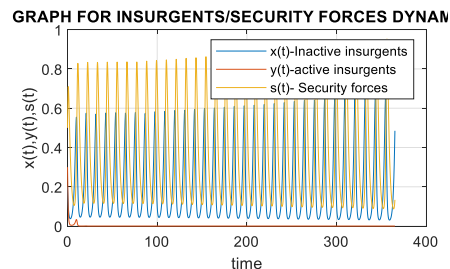


Figure BB1: Simulation Result for the model with –ve variations in  $\phi, \mu, r_1, r_2$  and  $\phi$  as contained in column BB2 in Table C4.

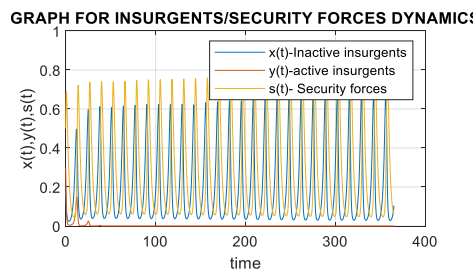


Figure BB2: Simulation Result for the model with –ve variations in  $\phi, \mu, r_1, r_2$  and  $\phi$  as contained in column BB3 in Table C4.

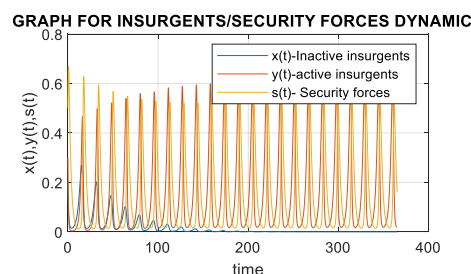


Figure BB3: Simulation Result for the model with –ve variations in  $\phi, \mu, r_1, r_2$  and  $\phi$  as contained in column BB4 in Table C4.

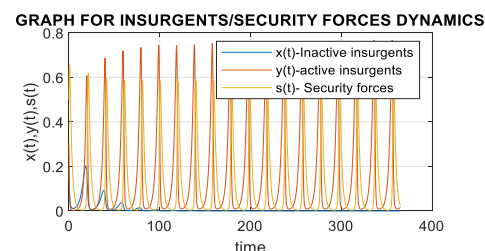


Figure BB4: Simulation Result for the model with –ve variations in  $\phi, \mu, r_1, r_2$  and  $\phi$  as contained in column BB5 in table C4.

**TableC1A: Baseline parameter values** A1 and variations in the rehabilitation rate  $\alpha$  used in numerical simulations Presented in Figures A1 to A3

| Parameter  | Figure A1 | Figure A2 | Figure A3 |
|------------|-----------|-----------|-----------|
| $r_1$      |           | 1.1       | 1.10      |
| $r_2$      |           | 1.40      | 1.40      |
| $\phi$     |           | 1.50      | 1.50      |
| $\alpha$   | 0.70      | 0.80      | 1.00      |
| $\beta$    | 0.80      | 0.80      | 0.80      |
| $\gamma_1$ | 1.00      | 1.00      | 1.00      |
| $\gamma_2$ | 2.10      | 2.10      | 2.10      |
| $\sigma$   | 2.10      | 2.10      | 2.10      |
| $\pi$      | 2.80      | 2.80      | 2.80      |
| $\varphi$  | 0.02      | 0.02      | 0.02      |
| $\mu$      | 0.001     |           | 0.001     |
| $d$        | 0.50      |           | 0.50      |

**TableC1B:** Baseline parameter values and variations in the rehabilitation rate  $\alpha$  used in numerical simulations Presented in Figures A4 and A5

| Parameter  | Figure A4 | Figure A5 |
|------------|-----------|-----------|
| $r_1$      |           | 1.10      |
| $r_2$      |           | 1.40      |
| $\phi$     |           | 1.50      |
| $\alpha$   |           | 1.40      |
| $\beta$    |           | 0.80      |
| $\gamma_1$ |           | 1.00      |
| $\gamma_2$ |           | 2.10      |
| $\sigma$   |           | 2.10      |
| $\pi$      |           | 2.80      |
| $\varphi$  |           | 0.02      |
| $\mu$      |           | 0.001     |
| $d$        |           | 0.50      |

**TableC2A:** Table of parameter values with positive varied values of  $\alpha, \beta, \gamma_1, \gamma_2, \sigma, \pi$  and varying negatively the values of  $r_1, r_2, \phi$  and  $\varphi$  during numerical simulations Presented in Figure A1,B1,B2

| Parameter | Figure A1 (Baseline) | Figure B1 | Figure B2 |
|-----------|----------------------|-----------|-----------|
| $r_1$     |                      | 1.10      | 1.05      |
| $r_2$     |                      | 1.40      | 1.20      |
| $\phi$    |                      | 1.50      | 1.00      |
| $\alpha$  |                      | 0.70      | 0.90      |

|            |      |       |        |        |
|------------|------|-------|--------|--------|
| $\beta$    |      | 0.80  |        | 0.85   |
|            | 0.90 |       |        |        |
| $\gamma_1$ |      | 1.00  |        | 1.10   |
|            | 1.15 |       |        |        |
| $\gamma_2$ |      | 2.10  |        | 2.40   |
|            | 2.45 |       |        |        |
| $\sigma$   |      | 2.10  |        | 2.20   |
|            | 2.30 |       |        |        |
| $\pi$      |      | 2.80  | 2.90   | 2.95   |
| $\varphi$  |      | 0.02  | 0.01   | 0.008  |
| $\mu$      |      | 0.001 | 0.0005 | 0.0003 |
| $d$        | 0.50 |       | 0.80   | 0.80   |

**TableC2B:** Table of parameter values with positive varied values of  $\alpha, \beta, \gamma_1, \gamma_2, \sigma, \pi$  and varying negatively the values of  $r_1, r_2, \phi$  and  $\phi$  during numerical simulations Presented in Figure B3 and B4

| Parameter  | Figure B3 | Figure B4 |
|------------|-----------|-----------|
| $r_1$      | 0.70      | 0.60      |
| $r_2$      | 0.70      | 0.50      |
| $\phi$     | 0.8       | 0.7       |
| $\alpha$   | 1.10      | 1.20      |
| $\beta$    | 1.10      | 1.30      |
| $\gamma_1$ | 1.25      | 2.30      |
| $\gamma_2$ | 2.50      | 2.55      |
| $\sigma$   | 2.50      | 3.0       |
| $\pi$      | 3.0       | 3.1       |
| $\phi$     | 0.9       | 0.9       |
| $\mu$      | 0.0002    | 0.0001    |
| $d$        | 0.80      | 0.31      |

**TableC3A:** Baseline parameter values with –ve variations in the values of  $\beta$  while keeping other parameters constant, and used in the numerical simulations Presented in Figure AA1 to AA3

| Parameter  | Figure AA1 | Figure AA2 | Figure AA3 |
|------------|------------|------------|------------|
| $r_1$      | 1.1        | 1.1        | 1.1        |
| $r_2$      | 1.40       | 1.40       | 1.40       |
| $\phi$     | 1.50       | 1.50       | 1.50       |
| $\alpha$   | 0.70       | 0.80       | 1.0        |
| $\beta$    | 0.60       | 0.70       | 0.80       |
| $\gamma_1$ | 1.00       | 1.00       | 1.00       |
| $\gamma_2$ | 2.10       | 2.10       | 2.10       |
| $\sigma$   | 2.10       | 2.10       | 2.10       |
| $\pi$      | 2.80       | 2.80       | 2.80       |
| $\phi$     | 0.02       | 0.02       | 0.02       |
| $\mu$      | 0.001      | 0.001      | 0.001      |
| $d$        | 0.50       | 0.50       | 0.50       |

**TableC3B:** Baseline parameter values with –ve variations in the values of  $\beta$  while keeping other parameters constant, and used in the numerical simulations Presented in Figure AA4 to AA5

| Parameter  | Figure AA4 | Figure AA5 |       |
|------------|------------|------------|-------|
| $r_1$      |            | 1.1        | 1.1   |
| $r_2$      |            | 1.40       | 1.40  |
| $\phi$     |            | 1.50       | 1.50  |
| $\alpha$   |            | 1.2        | 1.4   |
| $\beta$    |            | 0.90       | 1.0   |
| $\gamma_1$ |            | 1.00       | 1.00  |
| $\gamma_2$ |            | 2.10       | 2.10  |
| $\sigma$   |            | 2.10       | 2.10  |
| $\pi$      |            | 2.80       | 2.80  |
| $\varphi$  |            | 0.02       | 0.02  |
| $\mu$      |            | 0.001      | 0.001 |
| $d$        |            | 0.50       | 0.50  |

**TableC4A:** Baseline parameter values with –ve variations in  $\varphi, \mu, r_1, r_2$  and  $\phi$  used in numerical simulations Presented in Figure BB1 TO BB2

| Parameter  | Figure BB1 | FigureBB2 |        |
|------------|------------|-----------|--------|
| $r_1$      |            | 0.8       |        |
| $r_2$      | 0.60       | 0.7       | 0.60   |
| $\phi$     |            | 1.0       |        |
| $\alpha$   | 0.90       | 0.80      | 0.80   |
| $\beta$    |            | 0.80      | 0.80   |
| $\gamma_1$ |            | 1.00      | 1.00   |
| $\gamma_2$ |            | 2.10      | 2.10   |
| $\sigma$   |            | 2.40      | 2.10   |
| $\pi$      |            | 2.80      | 2.80   |
| $\varphi$  |            | 0.007     | 0.006  |
| $\mu$      |            | 0.0003    | 0.0002 |
| $d$        |            | 0.50      | 0.50   |

**TableC4B:** Baseline parameter values with –ve variations in  $\varphi, \mu, r_1, r_2$  and  $\phi$  used in numerical simulations Presented in Figure BB3 TO BB4

| Parameter  | Figure BB3 | Figure BB4 |
|------------|------------|------------|
| $r_1$      |            | 0.40       |
| 0.30       |            |            |
| $r_2$      | 0.50       |            |
| 0.40       |            |            |
| $\phi$     |            | 0.70       |
| 0.60       |            |            |
| $\alpha$   | 0.80       |            |
| 0.80       |            |            |
| $\beta$    | 0.80       |            |
| 0.80       |            |            |
| $\gamma_1$ | 1.00       |            |
| 1.00       |            |            |
| $\gamma_2$ | 2.10       |            |
| 2.10       |            |            |
| $\sigma$   | 2.10       |            |
| 2.10       |            |            |
| $\pi$      | 2.80       |            |
| 2.80       |            |            |
| $\varphi$  | 0.005      | 0.003      |
| $\mu$      | 0.0001     | 0.00008    |
| $d$        | 0.50       |            |
| 0.50       |            |            |

ACKNOWLEDGMENT

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