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Solution of linear ordinary differential equations in a fuzzy environment by method of variation of parameters

Udot Effiong Asuquo[1], Alhaji Tahir[2*], Usman Sulaiman[2]

[1]Mathematics Department, Federal Government Girls College, Yola, Nigeria

[2]Department of Mathematics, Modibbo Adama University, Yola, Nigeria

ABSTRACT

This work considers linear fuzzy initial value problems (LFIVPs) defined under gH-differentiability with the aim of establishing a method and, subsequently, a fuzzy-valued function, which is a solution to the LFIVPs. The conditions for a fuzzy function to be Hukuhara (H)-differentiable and gH-differentiability are defined. It is, however, established that the gH-differentiability concept to fuzzy-valued functions is more generalized than H-differentiable. Characterization and stacking theorems are also used to translate LFIVPs to a system of specialized ODEs. The method of variation of parameters is also extended to LFIVPs. Examples are constructed to test the applicability of the methods and graphical presentation of the behavior of solutions of the constructed examples is carried out with the aid of MATLAB version 8.5. We recommend that further studies should be carried out on second and higher order LFIVPs.

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The idea of fuzzy systems was initially viewed as systems whose state equations involved fuzzy variables or parameters, giving birth to fuzzy classes of systems. Another idea was that a system is fuzzy if its input, output, or state would range over a family of fuzzy sets. Zadeh (1971) suggested that a fuzzy system could be a generalization of a non-deterministic system that moves from a state to a fuzzy set of states. Fuzzy differential equations (FDEs) appeared as a natural way to model the propagation of scientific uncertainty in a dynamical environment. There are several interpretations of FDEs. Basically, the principal disadvantage of Hukuhara (H)-derivative interpretation lies in the fact that it precludes the existence of periodic solutions on asymptotic phenomena. That is why different ideas and methods to solve FDEs have been developed. One of the methods solves differential equations using Zadeh's extension principle, while another interprets FDEs through differential inclusions.

A more generalized concept of differentiability extends the previous concepts of H-differentiable and (ii) gH-differentiability, respectively, and allows one to choose (i) H-differentiable and (ii) gH-differentiability, which was also introduced. The approach allows one to solve some fuzzy differential inclusions for zero-level sets and find the nature of the support of the solution, i.e., either the solution is increasing or decreasing. On the interval where the solution is increasing, (i) H-differentiable is chosen and on the interval where the solution is decreasing and (ii) gH-differentiability is, therefore, considered (Bede and Gal, 2004). Pirzada and Vakaskar (2016) worked extensively on

*** Corresponding author** Alhaji Tahir ✉ atahir@mautech.edu.ng Department of Mathematics, Modibbo Adama University, Yola, Nigeria.

H-differentiable of a fuzzy-valued function and the H difference of two fuzzy sets $\tilde{a}$ and $\tilde{b}$, $\tilde{a}\,\text{Ä}_H\,\tilde{b}$, was established. Bede (2013) compared H-differentiable function with Seikkala derivative. He stated that a fuzzy number $f(x)\in R_{\tilde{F}}$ is a Seikkala derivative of a fuzzy number-valued function $f:(a,b)\to R_{\tilde{F}}$ if it is defined by $[f'(x)]^r=\left[\left(\underline{f}^r(x)\right)',\left(\overline{f}^r(x)\right)'\right]$, $0\le r\le 1$.

Others works include that of Bede and Stefanini (2011) who used LU-parametric representation by means of the Negoita–Ralescu representation theorem to solve FDEs, based on a strong gH-differentiability. Rudas and Bencsik (2006) used the method of variation of constants for FDEs with strong generalized differentiability. See also Bede (2007), Mondal et al. (2016), Bede and Gal (2010), Patel and Desai (2017), Bede and Stefanini (2009), Tapaswini and Chakraverty (2014), and Ghanbari (2012) who used different approaches such as the characterization theorem, fuzzy Laplace transform, variational iterative method, and Adomian decomposition method to establish results for FDEs. Lastly, Bede (2013) generated a solution of fuzzy initial values using Zadeh's extension principle on the classical solution of ordinary differential equations (ODEs).

The aim of this work is to establish a more general approach and sufficiently gH-differentiable function $\eta(x)$ for the solution of FDEs. Therefore, gH-differentiability and characterization theorem are considered in order to proffer a fuzzy-valued solution of FDEs taking care of the gaps found in the work of Mondal et al. (2016). In their work, a non-homogenous linear fuzzy initial value problem (LFIVP) was considered, and a solution under H-derivative was proposed, but it was later claimed that the solution is H-differentiable. It is, however, observed that their solution was short of the requirements for a function to be H-differentiable, which coincides with a case of (i) H-differentiable. Moreover, the H-differentiable function must have an increasing length of support, i.e., if a function $u(x)$ is H-differentiable, then $u'(x)\ge 0$ also $u(x)=-u(x)=u'(x)$ (Pirzada and Vakaskar, 2016). Therefore, the conditions for H-differentiable mentioned above are lacking in the work of Mondal et al. (2016).

2. Methodology

This section presents the methods used in establishing a valued function $\eta(x)$ which is a solution to LFIVP.

2.1. Conditions of Hukahara differentiability

It is of great importance to consider the general form of FIVP due to Dinius (2017).

$$\begin{cases} x^n(t)=f\left(t,x(t),x'(t),x''(t),\ldots,x^{n-1}(t)\right) \\ x(t_0)=x_0 \end{cases} \tag{2.1}$$

where

(i) f is a fuzzy function;

(ii) $t\in 0,1$;

(iii) $x,x',x'',x''',\ldots,x^{n-1}$ are fuzzy derivatives of x;

(iv) $x_0=(\underline{x}_0,\overline{x}_0)$ is a fuzzy number.

Note that conditions (i)–(iv) must satisfy the characterization theorem stated below.

Lemma 2.1 (Characterization theorem; Bede et al., 2007): Let $f:(a,b)\to R_{\tilde{F}}$ be H- differentiable and denote $[f(t)]^r=[\underline{f}^r(t),\overline{f}^r(t)]$ Then, the r-level boundary functions $\underline{f}^r$ and $\overline{f}^r$ are differentiable and

$$[f'(t)]^r=\left[\left(\underline{f}^r(t)\right)',\left(\overline{f}^r(t)\right)'\right],\ r\in[0,1].$$

Considering the FIVP (2.1) with $n=1$, we have

$$\begin{cases} x'=f(t,x) \\ x(t_0)=x_0 \end{cases} \tag{2.2}$$

where $f:[t_0,t_0+a]\times R_{\tilde{F}}\to R_{\tilde{F}}, x_0\in R_{\tilde{F}}$,and $[f(t,x)]^r=\left[\underline{f}^r\left(t,\underline{x}^r,\overline{x}^r\right),\overline{f}^r\left(t,\underline{x}^r,\overline{x}^r\right)\right]$.
Let $[x(t)]^r=\left[\underline{x}^r(t),\overline{x}^r(t)\right]$, if $x(t)$ is H-differentiable, then $[x'(t)]^r=\left[\left(\underline{x}^r(t)\right)',\left(\overline{x}^r(t)\right)'\right]$.

Equation (2.2) translates to

$$\begin{cases} \left(\underline{x}^r(t)\right)'=\overline{f}^r\left(t,\underline{x}^r(t),\overline{x}^r(t)\right), \\ \left(\overline{x}^r(t)\right)'=\underline{f}^r\left(t,\underline{x}^r(t),\overline{x}^r(t)\right), \\ \left(\underline{x}^r(t)\right)'=\underline{f}^r\left(t,\underline{x}^r(t),\overline{x}^r(t)\right), \\ \left(\overline{x}^r(t)\right)'=\overline{f}^r\left(t,\underline{x}^r(t),\overline{x}^r(t)\right), \\ \underline{x}^r(0)=(\underline{x}_0)^r,\overline{x}^r(0)=(\overline{x}_0)^r \end{cases} \tag{2.3}$$

If x_0 is a triangular fuzzy number, i.e., $x_0=\left(\underline{x}_0,x_0^1,\overline{x}_0\right)\in R_{\tilde{F}}$, $x(t)=\left(\underline{x},x^1,\overline{x}\right)\in R_{\tilde{F}}$, $f:[t_0,t_0+a]\times R_{\tilde{F}}\to R_{\tilde{F}}$, *and* $f\left(t,\underline{x},x^{(1)},\overline{x}\right)=\left\{\underline{f}\left(t,\underline{x},x^{(1)},\overline{x}\right),f^{(1)}\left(t,\underline{x},x^{(1)},\overline{x}\right),\overline{f}\left(t,\underline{x},x^{(1)},\overline{x}\right)\right\}$.

Translating Equation (2.2) into the system of ODEs yields

$$\begin{cases} \left(\underline{x}^r(t)\right)'=\overline{f}^r\left(t,\underline{x}^r(t),x^{(1)r}(t),\overline{x}^r(t)\right), \\ \left(\overline{x}^r(t)\right)'=\underline{f}^r\left(t,\underline{x}^r(t),x^{(1)r}(t),\overline{x}^r(t)\right), \\ \left(\underline{x}^r(t)\right)'=\underline{f}^r\left(t,\underline{x}^r(t),x^{(1)r}(t),\overline{x}^r(t)\right), \\ \left(\overline{x}^r(t)\right)'=\overline{f}^r\left(t,\underline{x}^r(t),x^{(1)^r}(t),\overline{x}^r(t)\right), \\ \left(x^{(1)^r}(t)\right)'=f^{(1)^r}\left(t,\underline{x}^r(t),x^{(1)^r}(t),\overline{x}^r(t)\right. \\ \underline{x}^r(0)=(\underline{x}_0)^r,x^{(1)^r}(0)=\left(x_0^{(1)}\right)^r,\overline{x}^r(0)= \end{cases} \tag{2.4}$$

The following definitions are as a result of Lemma 2.1.

Definition 2.1 (H-differentiable): Let $f:(a,b)\rightarrow R_{\tilde{f}}$ for h > 0 be sufficiently small, H-differences $f(x+h)\text{Ä}f(x)$ and $f(x)\text{Ä}f(x-h)$ exist, and there exists an element $f'(x)\in R_{\tilde{F}}$ such that $\lim_{h\to0}\frac{f(x+h)\text{Ä}f(x)}{h}=\lim_{h\to0}\frac{f(x)\text{Ä}f(x-h)}{h}=f'(x)$, then the function f is H-differentiable and the fuzzy derivative $f'(x)$ is called the H-derivative of the function f at x (Bede et al., 2007).

Definition 2.2 (gH-differentiability): Let $f:(a,b)\rightarrow R_{\tilde{F}}$ and $x_0\in(a,b)$; we can say that f is gH-differentiable if there exists an element $f'(x_0)\in R_{\tilde{F}}$ such that ∀ h > 0 is sufficiently small and there exists $f(x_0)\text{Ä}f(x_0+h)$, $f(x_0-h)\text{Ä}f(x_0)$, and the limits in the metric $(d_{\tilde{F}},R_{\tilde{F}})$ $\lim_{h\to0}\frac{f(x_0)\text{Ä}f(x_0+h)}{-h}=\lim_{h\to0}\frac{f(x_0-h)\text{Ä}f(x_0)}{-h}=f'(x_0)$ holds (Bede et al., 2007).

Definition 2.3 (Strong gH-differentiability): Let $f:(a,b)\rightarrow R_{\tilde{F}}$ and $x_0\in(a,b)$; we say that f is a strong gH-differentiable at x0, if there exists an element $f'(x_0)\in R_{\tilde{F}}$ such that

(i) $\forall h>0$ is sufficiently small, $\exists\ f(x_0+h)\text{Ä}f(x_0), f(x_0)\text{Ä}f(x_0-h)$, and the limits in the metric $(d_{\tilde{F}},R_{\tilde{F}})$; $\lim_{h\to0}\frac{f(x_0+h)\text{Ä}f(x_0)}{h}=\lim_{h\to0}\frac{f(x_0)\text{Ä}f(x_0-h)}{h}=f'(x_0)$ It should be noted that (i) H-differentiable coincides with (ii) gH-differentiability.

(ii) $\forall h>0$ is sufficiently small, $\exists f(x_0)\text{Ä}f(x_0+h), f(x_0-h)\text{Ä}f(x_0)$, and the limits. $\lim_{h\to0}\frac{f(x_0)\text{Ä}f(x_0+h)}{(-h)}=\lim_{h\to0}\frac{f(x_0-h)\text{Ä}f(x_0)}{(-h)}=f'(x_0)$.

(iii) □ h□0 is sufficiently small, $\exists f(x_0+h)\text{Ä}f(x_0), f(x_0-h)\text{Ä}f(x_0)$ and the limits $\lim_{h\to0}\frac{f(x_0+h)\text{Ä}f(x_0)}{h}=\lim_{h\to0}\frac{f(x_0-h)\text{Ä}f(x_0)}{-h}=f'(x_0)$. Or

(iv) $\forall h>0$ is sufficiently small, $\exists f(x_0)\text{Ä}f(x_0+h), f(x_0)\text{Ä}f(x_0-h)$, and the limits $\lim_{h\to0}\frac{f(x_0)\text{Ä}f(x_0+h)}{-h}=\lim_{h\to0}\frac{f(x_0)\text{Ä}f(x_0-h)}{h}=f'(x_0)$ (Bede and Stefanini, 2013).

It should be noted that when the derivative is not defined at x_0, then it is a weak gH-differentiable.

2.2. Translation of FIVP

From Equation (2.2), let $f:\ 0,T\ \times R_{\tilde{F}}\rightarrow R_{\tilde{F}}$ be a continuous fuzzy function,

where $f(t,x)$ is a fuzzy function of the crisp variable t, x is a fuzzy variable, x' is the fuzzy derivative of x, and $x(t_0)=x_0$ is a fuzzy number. Equation (2.2) is an FIVP of the first order (Cano et al., 2011). If $f:\ t_0,t_0+a\ \times R_{\tilde{F}}\rightarrow R_{\tilde{F}}$ and $x_0\in R_{\tilde{F}}$, then by Lemma 2.1 Equation (2.2) is translated into a system of ODEs in the following manner: Let

$$x(t)^{\ r} = \underline{x}^r(t),\overline{x}^r(t)\ ,$$

if $x(t)$ is H-differentiable, then

$$x(t)^{\ r} = \left(\underline{x}^r(t)\right),\left(\overline{x}^r(t)\right)\ .$$

Therefore, Equation (2.2) translates into system Equations (2.3).

To ensure that the system Equation (2.3) is equivalent to Equation (2.2), it is important to show that if the solution of system (2.3) is a valid set of a fuzzy valued function, which satisfies the Lipschitz condition, then it is a unique solution. The theorem below shows that the converse result holds as well and the FIVP (2.2) is equivalent to the system (2.3).

Lemma 2.2 (Stacking theorem; Kaleva, 2006): Consider the FIVP (2.2) such that

$$f:\ t_0,t_0+a\ \times R_{\tilde{F}}\rightarrow R_{\tilde{F}}$$

then the following conditions must be satisfied.

(i) $f(t,x)^{\ r} = \underline{f}^r(t,\underline{x}^r,\overline{x}^r),\overline{f}^r(t,\underline{x}^r,\overline{x}^r)$.

(ii) $\underline{f}^r$ and $\overline{f}^r$ are equicontinuous. That is to say for any $\varepsilon>0$ and any

$(t,x,y)\in\ t_0,t_0+a\ \times R^2$

we have $\left|\underline{f}^r(t,x,y)-\overline{f}^r(t,x,y)\right|<\varepsilon$

or

$\left|\overline{f}^r(t,x,y)-\underline{f}^r(t,x,y)\right|<\varepsilon\quad\forall r\in[0,1]$, whenever $\|(t,x_1,y_1)-(t,x,y)\|<\delta$ uniformly bound on any bounded set.

(iii) There exists $L>0$ such that

$\left|\underline{f}^r(t,x_1,y_1)-\overline{f}^r(t,x_2,y_2)\right|\le LMax\{|x-u|,|y-v|\}$

or

$\left|\overline{f}^r(t,x_1,y_1)-\underline{f}(t,x_2,y_2)\right|\le LMax\{|x-u|,|y-v|\}\ \forall r\in[0,1]$.

Therefore, as previously mentioned, Equation (2.2) is equivalent to system (2.3).

2.3. Method of variation of parameters

This method is applicable to a linear system of equations as seen in Kreyszig (2010), which can be expressed in the form

$$x'(t)=A(t)x+g, \tag{2.5}$$

and the solution of the homogeneous part of Equation (2.5) is given as

$$x^{(n)}=X(t)C. \tag{2.6}$$

The method of variation of parameters consists of replacing C in Equation (2.6) with a variable $U(t)$ in order to have

$$x^{(p)}=X(t)U(t). \tag{2.7}$$

Substituting Equation (2.7) into Equation (2.5) leads to the solution of Equation (2.5) and is given by

$$x(t)=X(t)C+X(t)\int_{t_0}^{t}X^{-1}(s)g(s)ds. \tag{2.8}$$

3. Results and Discussion

3.1. Fuzzification of FIVP

To obtain a more general form of Equation (2.2), a perturbation $b(t)$ is introduced here. Therefore, the new FIVP can be expressed as

$$\begin{cases} X'(t) = F(t, X, b(t)) \\ X(0) = X_0 \end{cases} \tag{3.1}$$

where F is a fuzzy function of a crisp variable t, X is a fuzzy variable, X_0 is a fuzzy number, and X' is a fuzzy derivative of X. To fuzzify LFIVP (3.1), the parameters of differential equations are made as an approximation (interval) instead of an exact. The following cases depend on the interpretation stated above:

Case I: By Lemma 2.1, we have the following equations:

$$\begin{cases} \left(\underline{x}^r\right)'(t) = \underline{f}^r\left(t, \underline{x}^r, \overline{x}^r, \underline{b}\right); \underline{x}^r(0) = \underline{x}_0{}^r \\ \left(\overline{x}^r\right)'(t) = \overline{f}^r\left(t, \underline{x}^r, \overline{x}^r, \overline{b}\right); \overline{x}^r(0) = \overline{x}_0{}^r \\ \left(\underline{x}^r\right)'(t) = \overline{f}^r\left(t, \underline{x}^r, \overline{x}^r, \underline{b}\right); \underline{x}^r(0) = \underline{x}_0^r \\ \left(\overline{x}^r\right)'(t) = \underline{f}^r\left(t, \underline{x}^r, \overline{x}^r, \overline{b}\right); \overline{x}^r(0) = \overline{x}_0{}^r \end{cases}. \tag{3.2}$$

Substituting $x(0) = \left(\underline{x}_0, x_0{}^{(1)}, \overline{x}_0\right)$ into system (3.2) yields

$$\begin{cases} \left(\underline{x}^r\right)'(t) = \underline{f}^r\left(t, \underline{x}^r, x^{r(1)}, \overline{x}^r, \underline{b}\right); \underline{x}^r(o) = \underline{x}_0{}^r \\ x^{r(1)\prime}(t) = f^{r(1)}\left(t, \underline{x}^r, x^{r(1)}, \overline{x}^r, b^{(1)}\right); x^{r(1)}(0) = x_0{}^{r(1)} \\ \left(\overline{x}^r\right)'(t) = \overline{f}^r\left(t, \underline{x}^r, x^{r(1)}, \overline{x}^r, \overline{b}\right); \overline{x}^r(0) = \overline{x}_0{}^r \\ \left(\underline{x}^r\right)'(t) = \overline{f}^r\left(t, \underline{x}^r, x^{r(1)}, \overline{x}^r, \underline{b}\right); \underline{x}^r(0) = \underline{x}_0{}^r \\ \left(\overline{x}^r\right)'(t) = \underline{f}^r\left(t, \underline{x}^r, x^{r(1)}, \overline{x}^r, \overline{b}\right); \overline{x}^r(0) = \overline{x}_0{}^r \end{cases}. \tag{3.3}$$

Case II: If $F(x)^r = \{a_1 + (a_2 - a_1)r\}, \{a_3 - (a_3 - a_2)r\}$ for a triangular fuzzy number (a_1, a_2, a_3) such that $a_1 < a_2 < a_3$ and $r \in 0,1$, then we have $\underline{x}^r(0) = a_1 + (a_2 - a_1)r$ and $\overline{x}^r(0) = a_3 - (a_3 - a_2)r$. These can lead to the following system of ODEs:

$$\begin{cases} \left(\underline{x}^r\right)'(t) = \underline{f}^r\left(t, \underline{x}^r, \overline{x}^r, \underline{b}\right); \underline{x}^r(0) = a_1 + (a_2 - a_1)r \\ \left(\overline{x}^r\right)'(t) = \overline{f}^r\left(t, \underline{x}^r, \overline{x}^r, \overline{b}\right); \overline{x}^r(0) = a_3 - (a_3 - a_2)r \\ \left(\underline{x}^r\right)'(t) = \overline{f}^r\left(t, \underline{x}^r, \overline{x}^r, \underline{b}\right); \underline{x}^r(0) = a_1 + (a_2 - a_1)r \\ \left(\overline{x}^r\right)'(t) = \underline{f}^r\left(t, \underline{x}^r, \overline{x}^r, \overline{b}\right); \overline{x}^r(0) = a_3 - (a_3 - a_2)r \end{cases}. \tag{3.4}$$

The systems of ODEs (3.2), (3.3), and (3.4), although not equivalent, are different interpretations of Equation (3.1). Solving systems (3.2), (3.3), and (3.4) above, one needs two equations of the systems. This means that for solutions

$\eta(x) = \left(\underline{x}(t), \overline{x}(t)\right)$ and $\eta(x) = \left(\underline{x}(t), x^{(1)}(t), \overline{x}(t)\right)$

where $x(0) = \left(\underline{x}_0, \overline{x}_0\right)$ and $x(0) = \left(\underline{x}_0, x_0{}^{(1)}, \overline{x}_0\right)$ are considered, respectively.

Case III: Consider Equation (3.1) where $b(t) = 0$. The solution of Equation (3.1) is assumed to be $x(t) = x_0 e^{at}$, which implies that

$x(t) = (\beta_1, \beta_2, \beta_3)e^{(\beta_1, \beta_2, \beta_3)}$, $\beta_1 < \beta_2 < \beta_3$, $\eta(x) = \left(\underline{x}(t), \overline{x}(t)\right)$, and

$$\begin{cases} \underline{x}(t) = \left(\beta_1 + (\beta_2 - \beta_1)r\right)e^{(\beta_3 - (\beta_3 - \beta_2)r)t} \\ \overline{x}^r(t) = \left(\beta_3 - (\beta_3 - \beta_2)r\right)e^{t(\beta_1 + (\beta_2 - \beta_1)r)t} \end{cases}. \tag{3.5}$$

$\eta(x) = \left(\underline{x}, \overline{x}\right)$ is, therefore, considered for the upper-bound variable and lower-bound variable of system (3.1) with $b(t) = 0$, respectively.

3.2. Extension of method of variation of parameters to LFIVP

The system of Equations (3.2)–(3.4) is expressed in form of matrix. Thus,

$$X' = A(t)X + \ell(t) \tag{3.6}$$

where X' is a column vector of the derivatives $\underline{x}'$ and $\overline{x}'$, $A(t)$ n×n square matrix, X is a column vector of the variables $\underline{x}$ and $\overline{x}$, and $\ell(t)$ is a column vector of $\underline{b}(t)$ and $\overline{b}(t)$. The solution to Equation (3.6) is given as

$$\eta(x) = x^{(h)} + x^{(p)} \tag{3.7}$$

where

$$x^{(h)} = c_1 x^1 + \ldots + c_n x^{(n)}. \tag{3.8}$$

Equation (3.8) is considered to be solution of the homogeneous system on interval I. Next is transforming Equation (3.8) into

$$x^{(h)} = \begin{bmatrix} c_1 x_1{}^{(1)} & + \ldots + & c_n x_1{}^{(n)} \\ \vdots & \vdots & \vdots \\ c_1 x_n{}^{(1)} & + \ldots + & c_n x_n{}^{(n)} \end{bmatrix} = \begin{bmatrix} x_1{}^{(1)} & \ldots & x_1{}^{(n)} \\ \vdots & \vdots & \vdots \\ x_n{}^{(1)} & \ldots & x_n{}^{(n)} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \tag{3.9}$$

to obtain

$$x^{(h)}(t) = J(t)C \tag{3.10}$$

where $J(t)$ is the matrix with column $x^{(1)}(t), \ldots, x^{(n)}(t)$, and $c^T = c_1 \ldots c_n$ is constant. The method of variation of parameters allows the replacement of the constant vector C with a variable $U(t)$ in Equation (3.10), which gives

$$x^{(p)} = J(t)U(t). \tag{13.11}$$

Also, to determine $U(t)$, substitute $x^{(p)}$ into Equation (3.6) and have

$$J'U + JU' = AJU + \ell. \tag{3.12}$$

By the solution of the matrix in Equation (3.6), we mean $X^{(1)'} = AX^{(1)},\ X^{(2)'} = AX^{(2)},\dots,X^{(n)'} = AX^{(n)}$. These n vector equations can be written as a single matrix equation $J' = AJ$.

Hence, Equation (3.12) is reduced to $JU' = \ell(t)$. Now J is nonsingular because n solutions of a basis are linearly independent. Hence, J^{-1} exists and we can solve $JU' = \ell(t)$, therefore,

$$U' = J^{-1}\ell(t). \tag{3.13}$$

Integrating Equation (3.13) from t_0 to t at some interval I, we obtain

$$U(t) = \int_{t_0}^{t} J^{-1}(s)\ell(s)ds. \tag{3.14}$$

Adding Equations (3.10) and (3.11), we obtain the general solution in form of

$$X(t) = J(t)C + J(t)\int_{t_0}^{t} J^{-1}(s)\ell(s)ds. \tag{3.15}$$

Set $\ell(t) = 0$ and substituting into Equation (3.6), we have

$$X'(t) = A(t)X. \tag{3.16}$$

Therefore,

$$X(t) = J(t)C. \tag{3.17}$$

The functions $\eta(x) = X(t)$ in both Equations (3.15) and (3.17) are solutions of Equation (3.6) in non-homogeneous system when $\ell(t) \neq 0$ and homogeneous system when $\ell(t) = 0$, respectively.

3.3. Translation of FIVP into a system of specialized ODEs

Two distinct methods can be applied directly to Equation (3.1) to obtain $\eta(x)$. Therefore, we came up with the following definitions that are used as a basis to obtain the new methods and a fuzzy valued function $\eta(x)$ as well.

Definition 3.1: Let $F:[0,T]\times R_{\tilde{F}} \to R_{\tilde{F}}$ be a continuous fuzzy function. Let $x_0 \in R_{\tilde{F}}$, where X' denotes a generalized derivative of a fuzzy-valued function X and $t\in[0,1]$, then Equation (3.1) is called a non-homogeneous FIVP and $b(t)$ is a perturbation.

Definition 3.2: A solution $\eta(x)$ which is a fuzzy-valued function for Equation (3.1) is a H-differentiable function $\eta(x):[0,T]\to R_{\tilde{F}}$ which satisfies Equation (3.1) for all $t\in[0,T]$.

Based on definitions (3.1) and (3.2), respectively, and the fuzzification of the ODEs above, we rewrite Equation (3.1) in the following forms:

(i) For $(x(0))' = (\underline{x}_0^r, \overline{x}_0^r)$, we consider

$$\begin{cases} (\underline{x}^r)'(t) = \overline{f}^r(t,\underline{x}^r(t),\overline{x}^r(t),\underline{b}(t)); \underline{x}^r(0) = \underline{x}_0^r \\ (\overline{x}^r)'(t) = \underline{f}^r(t,\underline{x}^r(t),\overline{x}^r(t),\overline{b}(t)); \overline{x}^r(0) = \overline{x}_0^r \end{cases} \tag{3.18}$$

where $\underline{x}^r$ and $\overline{x}^r$ are variables for each $r\in[0,1]$ and $b(t)$ is a perturbation.

(ii) For $(x(0))^r = (\underline{x}_0^r, x_0^{(1)}, \overline{x}_0^r)$, which incorporates a core variable $x_0^{(1)r}$, we consider

$$\begin{cases} (\underline{x})'(t) = \overline{f}^r(t,\underline{x}^r(t),x^{(1)r}(t),\overline{x}^r(t),\underline{b}(t)); \underline{x}^r(0) = \underline{x}_0^r \\ (x^{(1)r})'(t) = f^{(1)r}(t,\underline{x}^r(t),x^{(1)r}(t),\overline{x}^r(t),b^{(1)}(t)); x^{(1)r}(0) = x_0^{(1)r} \\ (\overline{x}^r)'(t) = \overline{f}^r(t,\underline{x}^r(t),x^{(1)r}(t),\overline{x}^r(t),\overline{b}(t)); \overline{x}^r(0) = \overline{x}_0^r \end{cases} \tag{3.19}$$

where $\underline{x}^r, \overline{x}^r$, and $x^{(1)r}$ (the core) are variables for each $r\in[0,1]$ and $\tilde{b}(t)$ is a perturbation.

If $b(t) = 0$ in Equation (3.1), then the equation is similar to Equation (2.2) and it is called homogeneous LFIVP and the resulting ODEs is of the form of system (2.3), for $(x(0))^r = (\underline{x}_0^r, x_0^r)$, and in addition to that, the system

$$\begin{cases} (\underline{x}^r(t))' = \underline{f}^r\ \ t,\underline{x}^r(t),x^{(1)^r}(t),\overline{x}^r(t) \\ (x^{(1)}(t))' = f^{(1)^r}\ \ t,\underline{x}^r(t),x^{(1)^r}(t),\overline{x}^r(t) \\ (\overline{x}^r(t))' = \overline{f}^r\ \ t,\underline{x}^r(t),x^{(1)^r}(t),\overline{x}^r(t) \end{cases} \tag{3.20}$$

is for $(x(0))^r = (\underline{x}_0^r, x_0^{(1)}, \overline{x}_0^r)$ where $\underline{x}^r, \overline{x}^r$, and $x^{(1)r}$ are variables for each $r\in[0,1]$.

3.4. Extension of the method of variation of parameters for solving FDEs incorporating core variables

Based on the fuzzification and method of variation of parameters for the LFIVP, the following propositions are established for the solutions of LFIVPs.

Proposition 3.1 (non-homogeneous case of solving LFIVP): Given a perturbed LFIVP below

$$\begin{cases} x'(t) = a \odot x(t) \oplus b(t) \\ x(0) = x_0 \end{cases}. \tag{3.21}$$

If x is gH-differentiability and x_0 is a fuzzy number, then there exists a fuzzy-valued function $\eta(x)=\left(\underline{x}^r(t),\overline{x}^r(t)\right)$ or $\underline{x}^r(t),x^{(1)^r}(t),\overline{x}^r(t)$, which satisfies (3.21), where

$$\begin{cases} \underline{x}(t) = \frac{1}{2}(\underline{x}_0 + \overline{x}_0)e^{at} + \frac{1}{2}(\underline{x}_0 - \overline{x}_0)e^{-at} + \underline{D} \\ \overline{x}(t) = \frac{1}{2}(\underline{x}_0 + \overline{x}_0)e^{at} + \frac{1}{2}(\overline{x}_0 - \underline{x}_0)e^{-at} + \overline{D} \end{cases}$$

$a>0,\ a<0$, and the core is given by $x^{(1)}(t)=x_0^{(1)}e^{at}+D^{(1)}$ in the case of triangular fuzzy number where $\tilde{D}$ is given by as defined in Equation (3.14).

Proof: For Equation (3.21) to be defined under gH-differentiability, it has to be equivalent to the system of specialized ODEs below.

$$\begin{cases} \underline{x}'(t) = a \odot \overline{x}(t) \oplus \underline{b}(t); \underline{x}(0) = \underline{x}_0 \\ \overline{x}'(t) = a \odot \underline{x}(t) \oplus \overline{b}(t); \overline{x}(0) = \overline{x}_0 \end{cases}. \tag{3.22}$$

By section 3.2, Equation (3.22) can be rewritten as

$$\begin{pmatrix} \underline{x}'(t) \\ \overline{x}'(t) \end{pmatrix} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix} + \begin{pmatrix} \underline{b}(t) \\ \overline{b}(t) \end{pmatrix}, \tag{3.23}$$

which is expressed below as a polynomial of the form

$$X'(t) = A(t) + \ell(t). \tag{3.24}$$

Also as indicated in Section 3.2, from Equation (3.15), we have

$$\eta(x) = JC + \int_{t_0}^{t} J^{-1}(s)\ell(s)ds.$$

To find the characteristic equation, let $|A-\lambda I|=0$, for $a>0$, arbitrarily chosen, $a=\alpha$ and setting $\begin{vmatrix} -\lambda & \alpha \\ \alpha & -\lambda \end{vmatrix}=0,$ the characteristic equation is, therefore, obtained as

$$\lambda^2 - \alpha^2 = 0. \tag{3.25}$$

The eigenvalues are $\lambda_1 = \alpha$ and $\lambda_2 = -\alpha$. The eigenfunctions form $[A-\lambda I][\tilde{V}]=0$. For $\lambda_1=\alpha$, $\begin{bmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and corresponding eigenfunction is $\tilde{V}_1 = \begin{matrix} 1 \\ 1 \end{matrix}$.

Also, for $\lambda_2=-\alpha$, and $\begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$corresponding eigenfunction is $\tilde{V}_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Next is to find the fundamental matrix where

$$J(t) = \begin{bmatrix} e^{\alpha t} & e^{-\alpha t} \\ e^{\alpha t} & -e^{-\alpha t} \end{bmatrix}. \tag{3.26}$$

Since $J(t)$ is nonsingular, the inverse

$$J^{-1}(t) = \begin{bmatrix} \frac{1}{2}e^{-\alpha t} & \frac{1}{2}e^{-\alpha t} \\ \frac{1}{2}e^{\alpha t} & -\frac{1}{2}e^{\alpha t} \end{bmatrix}$$

exists and the solution is obtained as follows:

$$\eta(x) = \begin{bmatrix} c_1e^{\alpha t} + c_2e^{-\alpha t} \\ c_1e^{\alpha t} - c_2e^{-\alpha t} \end{bmatrix} + \tilde{D}, \tag{3.27}$$

$$\begin{cases} \underline{x}(t) = c_1e^{\alpha t} + c_2e^{-\alpha t} + \underline{D} \\ \overline{x}(t) = c_1e^{\alpha t} - c_2e^{-\alpha t} + \overline{D} \end{cases} \tag{3.28}$$

Where $\tilde{D}=J\int_{t_0}^{t}J^{-1}(s)\ell(s)ds$. Applying the initial condition $x(0)=(\underline{x}_0,\overline{x}_0)$ on Equation (3.28), therefore, $c_1=\frac{1}{2}(\underline{x}_0+\overline{x}_0)$ and $c_2=\frac{1}{2}(\underline{x}_0-\overline{x}_0)$ for $a>0$.

Now, if $a<0$, arbitrarily chosen $a=-\alpha$ and following the same procedure as in the case where $a=\alpha$ the following results are obtained.

$$J(t) = \begin{bmatrix} e^{\alpha t} & e^{-\alpha t} \\ -e^{\alpha t} & e^{-\alpha t} \end{bmatrix},\ J^{-1}(t) = \begin{bmatrix} \frac{1}{2}e^{-\alpha t} & -\frac{1}{2}e^{-\alpha t} \\ \frac{1}{2}e^{\alpha t} & \frac{1}{2}e^{\alpha t} \end{bmatrix}, \text{ and } \eta(x) = \begin{bmatrix} c_1e^{\alpha t} + c_2e^{-\alpha t} \\ -c_1e^{\alpha t} + c_2e^{-\alpha t} \end{bmatrix} + \tilde{D}.$$

Therefore,

$$\begin{cases} \underline{x}(t) = c_1e^{\alpha t} + c_2e^{-\alpha t} + \underline{D} \\ \overline{x}(t) = -c_1e^{\alpha t} + c_2e^{-\alpha t} + \overline{D} \end{cases} \tag{3.29}$$

where $\tilde{D}=J\int_{t_0}^{t}J^{-1}(s)\ell(s)ds$, as previously indicated above. Applying the initial condition on Equation (3.29), therefore, $c_1=\frac{1}{2}(\underline{x}_0-\overline{x}_0)$and $c_2=\frac{1}{2}(\underline{x}_0+\overline{x}_0)$ for $a<0$. Substituting the constants for $a>0$ and $a<0$ in Equations (3.28) and (3.29), respectively, we have

$$\begin{cases} \underline{x}(t) = \frac{1}{2}(\underline{x}_0 + \overline{x}_0)e^{at} + \frac{1}{2}(\underline{x}_0 - \overline{x}_0)e^{-at} + \underline{D} \\ \overline{x}(t) = \frac{1}{2}(\underline{x}_0 + \overline{x}_0)e^{at} + \frac{1}{2}(\overline{x}_0 - \underline{x}_0)e^{-at} + \overline{D} \end{cases}, \tag{3.30}$$

This satisfies both cases where $a<0$ and $a>0$, respectively. Hence, Equation (3.30) is the result established for solving non-homogeneous FIVP. This completes the proof. Consequently, as a result of Proposition 3.1 above, the following corollaries are true.

Corollary 3.1: In the case of triangular fuzzy number $x(0)=\left(\underline{x}_0,x_0^{(1)},\overline{x}_0\right)$, the core variable is given by $x^{(1)}(t)=x_0^{(1)}e^{at}+D^{(1)}$.

Proof: Let $x^{(1)'}(t)=ax^{(1)}(t)+b(t)$, we can rewrite this equation as $x^{(1)'}(t)-ax^{(1)}(t)=b(t)$. Let the integrating factor be e^{-at}, then we have

$x^{(1)'}(t)e^{-at} - ax^{(1)}(t)e^{-at} = b(t)e^{-at}$.

This gives

$\frac{d}{dt}\left(x^{(1)}(t)e^{-at}\right) = b(t)e^{-at}$.

Cross-multiplying and integrating the above equation we have,

$\int d\left(x^{(1)}(t)e^{-at}\right) = c + \int b(t)e^{-at}\,dt$

and this is equal to

$x^{(1)}(t) = ce^{at} + e^{at}\int b(t)e^{-at}\,dt$.

Let $e^{at}\int b(t)e^{-at}\,dt = k$ and $\left(e^{at}\int b(t)e^{-at}dt\right)|_{t=0} = k_0$, then we have. $x^{(1)}(0) = c + k_0$. But $x^{(1)}(0) = x_0^{(1)}$ so that we have $c = x_0^{(1)} - k_o$ and $x^{(1)}(t) = \left(x_0^{(1)} - k_0\right)e^{at} + k$, and $x^{(1)}(t) = x_0^{(1)}e^{at} + k - k_0e^{at}$. Let $D^{(1)} = K - K_0e^{at}$, then

$$x^{(1)}(t) = x_0^{(1)}e^{at} + D^{(1)}. \qquad (3.31)$$

The proof is complete.

Corollary 3.2: Evaluating $\tilde{D}$ at zero, equals to zero, i.e., $\tilde{D}_{t=0} = 0$.

Poof: For the sake of poof of corollary (3.2), we select some $\tilde{D}$ for $b(t)$ as follows:

(1) For algebraic function, $b(t) = nt^p$ where n is the coefficient and p is the exponent;

(i) $\tilde{D}_p = n\left\{\frac{p!t^0}{0!a^{p+1}}\left(e^{at}-1\right) - \sum_{i=1}^{p}\frac{p!t^i}{i!a^{a+1-i}}\right\}, a>0, p=0,1,2,\ldots i=1,2,3,\ldots$

(ii) $\tilde{D}_p = n\left\{g^p\frac{p!t^0}{0!(-a)^{p+1}}\left(1-e^{at}\right) + \sum_{i=1}^{p}g^{p+i}\frac{p!t^i}{i!(-a)^{p+1-i}}\right\}, a<0, p=0,1,2,\ldots, i=1,2,3,\ldots$

(2) For exponential function, $b(t) = ne^{pt}$ then

(i) $\tilde{D} = n\left\{\frac{1}{p-a}\left(e^{pt}-e^{at}\right)\right\}, a>0, p\geq 0$,

(ii) $\tilde{D} = nte^{pt}, a>0, a<0,\ p<0$ and $a = p$

(iii) $\tilde{D} = n\left\{\frac{1}{|a|+|p|}\left(e^{at}-e^{pt}\right)\right\}; a>0, p\leq 0$

(3) For trigonometric functions $b(t) = \cos(nt)$ and $b(t) = \sin(nt)$

(i) for $b(t) = \cos(nt)$, $\tilde{D} = \frac{a}{a^2+n^2}\left(e^{at}-\cos(nt)\right) + \frac{n\sin(nt)}{a^2+n^2}$

(ii) for $b(t) = \sin(nt)$, $\tilde{D} = \frac{n}{a^2+n^2}\left(e^{at}-\cos(nt)\right) - \frac{a\sin(nt)}{a^2+n^2}$.

With $\tilde{D}s$ above, if we substitute zero for t, we have $\tilde{D}_{t=0} = 0$ and this ends the proof.

The result obtained below is stated as Proposition 3.2, which is followed by its detailed proof.

Proposition 3.2 (homogeneous case for solving LFIVP): Given a homogeneous FIVP

$$\begin{cases} x'(t) = a\odot x \\ x(0) = x_0 \end{cases}. \qquad (3.32)$$

If x is gH-differentiability and x_0 is a fuzzy number, then there exists a fuzzy-valued function $\eta(x)\in R_{\tilde{F}}$ given by $\eta(x) = \left(\underline{x}^r(t), \overline{x}^r(t)\right)$ or $\left(\underline{x}(t)x^{(1)}(t), \overline{x}(t)\right)$ $\forall\ t\in[0,T], r\in[0,1]$ which satisfies the Equation (3.32), where

$$\begin{cases} \underline{x}^r(t) = \frac{1}{2}\left(\underline{x}_0+\overline{x}_0\right)e^{at} + \frac{1}{2}\left(\underline{x}_0-\overline{x}_0\right)e^{-at} \\ \overline{x}^r(t) = \frac{1}{2}\left(\underline{x}_0+\overline{x}_0\right)e^{at} + \frac{1}{2}\left(\overline{x}_0-\underline{x}_0\right)e^{-at} \end{cases}, \text{ for } a<0, a>0$$

In the case of $X(0) = \left(\underline{x}_0, x_0^{(1)}, \overline{x}_0\right)$, the core is given by $x^{(1)}(t) = x_0^{(1)}e^{at}$.

Proof: By Lemma 2.1, we write

$$\begin{cases} \underline{x}'(t) = a\odot\overline{x};\ \underline{x}(0) = \underline{x}_0 \\ \overline{x}'(t) = a\odot\underline{x};\ \overline{x}(0) = \overline{x}_0 \end{cases} \qquad (3.33)$$

Rewriting Equation (3.33) in matrix form gives

$$\begin{pmatrix} \underline{x}'(t) \\ \overline{x}'(t) \end{pmatrix} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix}$$

and in a polynomial form it is

$X'(t) = AX$

where $a > 0$, arbitrarily chosen $a = \alpha$

$J(t) = \begin{pmatrix} e^{\alpha t} & e^{-\alpha t} \\ e^{\alpha t} & -e^{-\alpha t} \end{pmatrix}$, $C = \begin{pmatrix} c_1 & c_2 \end{pmatrix}^T$, and $J(t)C = \begin{pmatrix} c_1e^{\alpha t}+c_2e^{-\alpha t} \\ c_1e^{\alpha t}-c_2e^{-\alpha t} \end{pmatrix}$,

then

$$\begin{cases} \underline{x}(t) = ce^{at} + ce^{-at} \\ \overline{x}(t) = c_1e^{at} - c_2e^{-at} \end{cases}, \qquad (3.35)$$

applying the initial condition $x(0) = \left(\underline{x}_0, \overline{x}_0\right)$ on Equation (3.35) results in

$c_1 = \frac{1}{2}\left(\underline{x}_0+\overline{x}_0\right)$ and $c_2 = \frac{1}{2}\left(\underline{x}_0-\overline{x}_0\right)$, for $a>0$.

Next, when $a<0$, arbitrarily chosen $a=-\alpha$, and following the same procedure as in the case where $a=\alpha$ is chosen above, we obtain

$A = \begin{bmatrix} 0 & -\alpha \\ -\alpha & 0 \end{bmatrix}$, $J(t) = \begin{bmatrix} e^{\alpha t} & e^{-\alpha t} \\ -e^{\alpha t} & e^{-\alpha t} \end{bmatrix}$ and $X(x) = JC = \begin{bmatrix} e^{\alpha t} & e^{-\alpha t} \\ -e^{\alpha t} & e^{-\alpha t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1e^{\alpha t}+c_2e^{-\alpha t} \\ -c_1e^{\alpha t}+c_2e^{-\alpha t} \end{bmatrix}$.

Therefore,

$$\begin{cases} \underline{x} = c_1 e^{\alpha t} + c_2 e^{-\alpha t} \\ \overline{x} = -c_1 e^{\alpha t} + c_2 e^{-\alpha t} \end{cases} \tag{3.36}$$

Considering that from Equation (3.33) $x(0) = (\underline{x}_0, \overline{x}_0)$ and applying it to Equation (3.36), we have $c_1 = \frac{1}{2}(\underline{x}_0 - \overline{x}_0)$ and $c_2 = \frac{1}{2}(\underline{x}_0 + \overline{x}_0)$, $a < 0$.

Substituting c_1, c_2 for $a > 0$ and $a < 0$ in Equations (3.35) and (3.36), respectively, we have

$$\begin{cases} \underline{x}(t) = \frac{1}{2}(\underline{x}_0 + \overline{x}_0)e^{at} + \frac{1}{2}(\underline{x}_0 - \overline{x}_0)e^{-at} \\ \overline{x}(t) = \frac{1}{2}(\underline{x}_0 + \overline{x}_0)e^{at} + \frac{1}{2}(\overline{x}_0 - \underline{x}_0)e^{-at} \end{cases}, \ a < 0, \ a > 0. \quad . \tag{3.37}$$

By corollary 3.1, $x^{(1)}(t) = x_0^{(1)} e^{at}$ for $(x(0))^r = (\underline{x}_0^r, x_0^{(1)}, \overline{x}_0^r)$ and this ends the proof.

3.5. Constructed examples

To test the applicability or otherwise of the results established in this study, two examples are constructed each for non-homogeneous and homogeneous cases, respectively.

Example 1: Solve the non-homogeneous LFIVP under gH-differentiability.

$$\begin{cases} x'(t) = -x + (-1,1)e^{-t} \\ x(0) = (-1,1) \end{cases}.$$

Solution: from Equation (3.2), this LFIVP is translated into

$$\begin{cases} \underline{x}' = -\overline{x} - e^{-t}; \underline{x}(0) = -1 \\ \overline{x}' = -\underline{x} + e^{-t}; \overline{x}(0) = 1 \end{cases}$$

and by Proposition 3.1

$$\begin{cases} \underline{x}(t) = \frac{1}{2}(\underline{x}_0 + \overline{x}_0)e^{at} + \frac{1}{2}(\underline{x}_0 - \overline{x}_0)e^{-at} + \underline{D} \\ \overline{x}(t) = \frac{1}{2}(\underline{x}_0 + \overline{x}_0)e^{at} + \frac{1}{2}(\overline{x}_0 - \underline{x}_0)e^{-at} + \overline{D} \end{cases},$$

$\underline{x}_0 = -1$, $\overline{x}_0 = 1$ and

$$\tilde{D} = \begin{bmatrix} e^t & e^{-t} \\ -e^t & e^{-t} \end{bmatrix} \int_0^t \begin{bmatrix} \frac{1}{2}e^{-s} & \frac{1}{2}e^{-s} \\ \frac{1}{2}e^{s} & \frac{1}{2}e^{s} \end{bmatrix} \begin{bmatrix} -e^{-s} \\ e^{-s} \end{bmatrix} ds.$$

Here

$\tilde{D} = \frac{1}{2}(e^t - e^{-t})$, $\underline{D} = \frac{1}{2}e^{-t} - \frac{1}{2}e^t$, $\overline{D} = \frac{1}{2}e^t - \frac{1}{2}e^{-t}$

and the solution is

$$\eta(x) = (\underline{x}(t), \overline{x}(t)) = \left(\frac{1}{2}e^{-t} - \frac{3}{2}e^t, \frac{3}{2}e^t - \frac{1}{2}e^{-t}\right). \tag{3.38}$$

Example 2: Solve the homogeneous LFIVP

$$\begin{cases} x'(t) = x(t) \\ x(0) = (2,3,4) \end{cases}.$$

Solution: Here, $a = 1$, $\underline{x}_0 = 2$, $\overline{x}_0 = 4$, and

$$\begin{cases} \underline{x}(t) = \frac{1}{2}(2+4)e^t + \frac{1}{2}(2-4)e^{-t} \\ x^{(1)}(t) = 3e^t \\ \overline{x}(t) = \frac{1}{2}(2+4)e^t + \frac{1}{2}(4-2)e^{-t} \end{cases}$$

and by Proposition 3.2

$$\begin{cases} \underline{x}(t) = 3e^t - e^{-t} \\ x^{(1)} = 3e^t \\ \overline{x}(t) = 3e^t + e^{-t} \end{cases}.$$

Therefore, the solution is

$$\eta(x) = (\underline{x}, x^{(1)}, \overline{x}) = \left\{(3e^t - e^{-t}), (3e^t), (3e^t + e^{-t})\right\}.$$

3.6. Graphical representation of the solutions of the constructed examples

This section deals with the diagrammatic representations of examples 1 and 2. Example 1 is for the non-homogeneous case, while example 2 is for the homogeneous case. Figure 1 shows a solution of example 1 and Figure 2 shows a solution for example 2. In addition to that, Table 1 presents the values for the solution of non-homogeneous LFIVP of example 1 given by Equation (3.36). The components of Table 1 are the crisp variables $t \in [0,1]$, $\frac{1}{2}e^{-t}$, and $\frac{3}{2}e^t$, the lower bound variable $\underline{x}$, and the upper bound variable $\overline{x}$. Table 2 represents a solution of homogeneous LFIVP of example 2 given by Equation (3.38). The components of Table 2 are the crisp variables $t \in [0,3]$, e^t, $3e^t$, e^{-t}, the lower bound variable $\underline{x}$, the core $x^{(1)}$, and the upper bound variable $\overline{x}$.

Table 1. Values of example 1.

t	$\frac{1}{2}e^{-t}$	$\frac{3}{2}e^t$	$\underline{x}$	$\overline{x}$
0.0	0.50	1.50	−1.00	1.00
0.2	0.41	1.83	−1.40	1.40
0.4	0.34	2.24	−1.90	1.90
0.6	0.27	2.73	−2.50	2.50
0.8	0.22	3.34	−3.10	3.10
1.0	0.18	4.04	−3.90	3.90

Table 2. Values of example 2.

t	e^t	$3e^t$	e^{-t}	$\underline{x}$	$x^{(1)}$	$\overline{x}$
0.0	1.00	3.00	1.00	2.00	3.00	4.00
0.5	1.65	5.00	0.61	4.40	5.00	5.60
1.0	2.72	8.20	0.37	7.80	8.20	8.60
1.5	4.48	13.50	0.22	13.30	13.50	13.70
2.0	7.39	22.20	0.14	22.10	22.20	22.30
2.5	12.18	36.60	0.08	36.50	36.60	36.70
3.0	20.09	60.30	0.05	60.30	60.30	60.40

3.5. Discussion of the results

The gH-differentiability concept and characterization theorem are used to translate FIVP into specialized ODEs. Both the non-homogeneous and homogeneous systems are considered separately in this work. The extended methods are easier to implement and give better solutions to problems regarding LFIVP when compared with most of the existing methods in the literature (Rudas and Bencsik, 2006; Ghanbari, 2017).

The examples constructed in this work are meant to test the applicability or otherwise the established propositions. Example 1 refers to the solution of the non-homogeneous form of LFIVP considered in Equation (3.1), while example 2 tests the conditions for the solution of homogeneous LFIVP. One can also observe that example 1 satisfies all the conditions of Proposition 3.1. Also, example 2 satisfies all the conditions of Proposition 3.2. The propositions can also be used conveniently to address problems regarding LFIVPs considered by Mondal et al. (2016).

Figure 1 shows that there is no change in the behavior of solution for the non-homogeneous LFIVP. Moreover, taking into cognizance the fact that the interpretation of example 1 is based on (ii) gH-differentiability, it shows that the solution exists and the relationship is satisfied. This further concludes that (ii) gH-differentiability coincides with (i) H-differentiable for a fuzzy function. In Figure 2, it is observed that the gap between the upper bound variable and lower bound variable decreases as t increases and eventually disappears. It is, therefore, deduced from the results obtained and the fuzzy function that the new methods are more generalized and very effective in solving LFIVPs, when compared with the work of Mondal et al. (2016). These methods are also distinct for non-homogeneous and homogeneous cases, respectively. However, it is observed that gH-differentiability is a more general approach to problems in FDEs as (ii) gH-differentiability coincides with (i) H-differentiability, and the solution to FDEs is sufficiently a fuzzy-valued function and serves as a convenient choice or approach to solving FDEs.

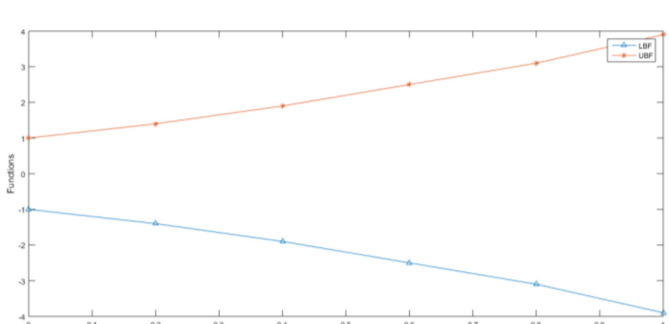

Figure 1. Graph of example 1.

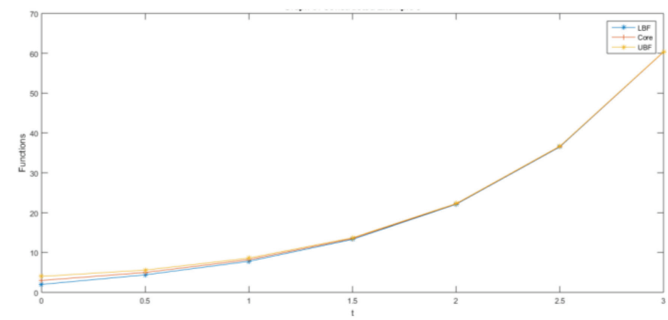

Figure 2. Graph of example 2.

4. Conclusion

Some FIVPs defined under gH-differentiability can be translated into a system of specialized ODEs. The application of various parameters induced two distinct methods for solving homogeneous and non-homogeneous linear FIVPs. The method of variation of parameters is extended and solutions of first-order linear FIVPs are obtained for the fuzzy-valued function.

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