https://doi.org/ 10.70882/aq1vrr32

SCIENCE FORUM (JOURNAL OF PURE AND APPLIED SCIENCES) 26, (4) (2026), 393 - 406

Science Forum (Journal of Pure and Applied Sciences)

Journal Homepage: https://atbuscienceforum.com.ng/index.php/jpas

Pw-closed Fuzzy Submodules and Pw-extending Fuzzy Modules

Hassan K. Marhon

Ministry of Education, General Directorate of Education of Baghdad Al-Resafa 1, Iraq.

Abstract

This paper develops a structural theory of pseudo w-closed (pw-closed) fuzzy submodules and the associated class of pw-extending fuzzy modules. The proposed condition extends the w-closed framework and is distinguished from related notions by inclusion results, examples, counterexamples, and correspondence theorems. Under condition (*), pw-closedness of a fuzzy submodule is equivalent to pw-closedness of its associated classical submodule, establishing a direct link between the fuzzy and classical settings. Further results concern direct sums, complete essentiality, full semi-primality, y-closedness, level constructions, and preservation under suitable surjective F-homomorphisms. The pw-extending class is then studied through its relations with w-extending and extending fuzzy modules, its inheritance under direct summands, and its connections with FI-extending and CLS-(F-)modules. Examples and counterexamples clarify the scope of the principal implications and distinguish the proposed class from neighboring notions. The results provide a structural basis for further investigations of generalized closedness and extending properties in fuzzy module theory.

Keywords

Pw-closed fuzzy submodules; pw-extending fuzzy modules; w-closed fuzzy submodules; w-extending fuzzy modules.

EISSN: 3156-5190 (Online)
ISSN: 1119-4618 (Print)
JPAS
Faculty of Science, ATBU Bauchi

Corresponding Author: Hassan K. Marhon, Ministry of Education, General Directorate of Education of Baghdad Al-Resafa1, Iraq.
Email: hassanmath316@gmail.com

1. Introduction

Fuzzy module theory provides a degree-valued setting in which classical module-theoretic structures can be reformulated and studied through their fuzzy counterparts. Essentiality, closedness, and extending conditions are particularly relevant because they describe the position of fuzzy submodules and their interaction with direct-summand decompositions.

A fuzzy submodule A of an R-module M, with ambient fuzzy module , X is called essential in X, denoted by $A \leq_e X$, when $A \cap B \neq 0_1$ for every non-trivial fuzzy submodule B of X. Equivalently, $A \cap B = 0_1$ implies $B = 0_1$ for every fuzzy submodule B of X [1]. A fuzzy submodule A is closed in X, written $A \leq c\ X$, when it has no proper essential extension; that is, $A \leq_e B \leq X$ implies $A = B$ [2].

A weaker form of essentiality is obtained through semi-prime F-submodules. A fuzzy submodule A is semi-prime if, for a fuzzy singleton r_t of R, a fuzzy point $a_s \subseteq X$, and k $\in \mathbb{Z}_+$, the relation $r_t{}^k a_s \subseteq A$ *implies* $r_t a_s \subseteq A$ [3]. Weak essentiality of $A \leq X$ is characterized by the condition $A \cap B \neq 0_1$ for every non-trivial semi-prime fuzzy submodule B of X [4]. Consequently, A is w-closed when it has no proper fuzzy submodule B of X in which A is weak essential [5].

The preceding notions motivate a weaker separation property. A fuzzy submodule A of X is called pseudo w-closed (pw-closed) when every fuzzy point x_t outside A can be separated from A by a w-closed fuzzy submodule B satisfying $A \subseteq B$ and $x_t \not\subset$B. Thus, every w-closed fuzzy submodule is pw-closed, whereas the converse fails in general. This distinction is important: pw-closedness enlarges the w-closed class while retaining a separation mechanism inherited from w-closedness.

The pw-closed condition leads naturally to a corresponding direct-summand theory. In the classical setting, a pseudo w-extending module is characterized by the requirement that every pw-closed submodule be a direct summand. We retain this defining feature in the fuzzy setting and investigate the resulting class in relation to w-extending and extending fuzzy modules.

The present work develops this framework in two stages. First, the structural theory of pw-closed fuzzy submodules is established. Under condition (*), the fuzzy property is characterized through the associated classical submodule. Examples are given to show strictness of the inclusion w-closed fuzzy submodules in the pw-closed class, the failure of heredity, finite-intersection closure, and the distinction from ordinary closedness. Further results concern direct sums, complete essentiality, fully semi-prime modules, y-closed submodules, and images under suitable F-epimorphisms.

Second, pw-extending fuzzy modules are investigated. The class is compared with w-extending and semi-simple fuzzy modules, and its relationship with extending fuzzy modules is determined under several natural hypotheses. In particular, fully semi-prime pw-extending fuzzy modules are shown to be stable under direct summands and to be extending; they also satisfy FI-extending and CLS properties. These results place the proposed class within the existing structure theory while exhibiting conditions under which the new notion yields stronger decomposition properties.

For brevity, the notation F-set, F-submodule and F-module will be used throughout fuzzy set, fuzzy submodule, and fuzzy module, respectively.

Fundamental concepts and properties

The basic concepts and properties required in the sequel include F-sets, F-modules, F-submodules, and their corresponding classical structures are described as follows.

Definition 1.1 [6]:

For a nonempty $\mathcal{S}$ an a F-subset is characterized by a mapping $\mu: \mathcal{S} \longrightarrow$ [0 1] .

Definition 1.2 [7]:

For any element $x \in \mathcal{S}$, $\ell \in [0,1]$, the F-set $x_\ell : \mathcal{S} \longrightarrow$ [0, 1] is the F-singleton defined by:

$$x_\ell = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y \end{cases} \qquad \forall \ y \in \mathcal{S}$$

In particular, when x = 0 and ℓ = 1 the F-singleton is given by :

$$0_1(y) = \begin{cases} 1 & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y \neq 0 \end{cases}$$

Definition 1.3 [7]:

For F-sets $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{S}$, their union and intersection are given by the following expressions, respectively by:

$(\mathcal{A} \cup \mathcal{B})(x)$ = max$\{(\mathcal{A}(x), \mathcal{B}(x)\}$, and$(\mathcal{A} \cap \mathcal{B})(x) = \min\{(\mathcal{A}(x), \mathcal{B}(x)\} \quad \forall\ x \in \mathcal{S}$.

Definition 1.4 [8]:

For an F-set $\mathcal{A}$ on $\mathcal{S}$, and t ∈ [0,1], its t-level set is denoted by $\mathcal{A}_t$= { $x \in \mathcal{S}$, $\mathcal{A}$ $(x) \geq$ t}.

Remark 1.5 [6]:

Given F-subsets $\mathcal{A}$and $\mathcal{B}$ of $\mathcal{S}$, and every t ∈ [0,1], the following hold:

1. $(\mathcal{A} \cap \mathcal{B})_t = \mathcal{A}_t \cap \mathcal{B}_t$.
2. $(\mathcal{A} \cup \mathcal{B})_t = \mathcal{A}_t \cup \mathcal{B}_t$.
3. $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A}_t = \mathcal{B}_t$.

Definition 1.6 [9]

An F-subset $\mathcal{A}$ associated with a mapping $f : \mathrm{M} \longrightarrow \mathrm{M}^{\sim}$ is called f-invariant whenever $\mathcal{A}$ (x) = $\mathcal{A}$ (y), implies $\mathcal{A}$(f (x)) = $\mathcal{A}$(f (y)), for all x, y ∈ M.

Proposition 1.7 [9]:

Consider a mapping f defined on a set M, together with F-subsets $\mathcal{A}_1$ *and* $\mathcal{A}_2$ of M $\mathcal{B}_1$ *and* $\mathcal{B}_2$ F-subsets of f (M). The following properties holds:

1. $\mathcal{A}_1 \subseteq f^{-1}(f(\mathcal{A}_1))$.
2. $\mathcal{A}_1 = f^{-1}(f(\mathcal{A}_1))$, whenever $\mathcal{A}_1$ is f-invariant.
3. $f(f^{-1}(\mathcal{B}_1)) = \mathcal{B}_1$.
4. If $\mathcal{A}_1 \subseteq \mathcal{A}_2$, then $f(\mathcal{A}_1) \subseteq f(\mathcal{A}_{2})$.
5. If $\mathcal{B}_1 \subseteq \mathcal{B}_2$,then $f^{-1}(\mathcal{B}_1) \subseteq f^{-1}(\mathcal{B}_2)$.

Definition 1.8 [6]:

An F-set $\mathcal{X}$ of an $\mathcal{R}-$module M is an F-module whenever:

1. $\mathcal{X}$ $(x-y) \geq$ min $\{\mathrm{X}(x), \mathrm{X}(y)\}$, ∀ x, y ∈. M
2. $\mathcal{X}$ (rχ) ≥ X (χ), ∀ χ ∈ M and r ∈ $\mathcal{R}$.
3. $\mathcal{X}$ (0) = 1.

Definition 1.9 [8]:

Given F-modules $\mathcal{A}$ and $\mathcal{X}$ over an $\mathcal{R}$-module M, $\mathcal{A}$ is an F-submodule of $\mathcal{X}$ when $\mathcal{A} \subseteq \mathcal{X}$ and $\mathcal{A}$ satisfies the F-module conditions.

Definition 1.10 [10]:

At each t ∈ [0,1], $\mathcal{A}_t$ is the t-level submodule of $\mathcal{X}_t$ induced by $\mathcal{A}$.

Definition 1.11 [11]:

An F-module X over an R-module M is called fully f-invariant whenever each F-submodule of $\mathcal{X}$ is f-invariant.

Definition 1.12 [5]:

The F-singular submodule Z($\mathcal{X}$) of $\mathcal{X}$, is given by: Z($\mathcal{X}$) = $\{x_t \subseteq \mathcal{X}$: F-ann$(x_t)$ is an essential F-ideal of $\mathcal{R}\}$. The condition Z($\mathcal{X}$) = $\mathcal{X}$ and characterizes $\mathcal{X}$ as F-singular, while Z($\mathcal{X}$) = 0_1 characterizes $\mathcal{X}$ as F-nonsingular.

Definition 1.13 [12]:

An F-module $\mathcal{X}$ over an R-module M is semi-simple if every F-submodules of $\mathcal{X}$ is a direct summand of $\mathcal{X}$.

Definition 1.14 [13]:

The F-module $\mathcal{X}$ is said to be fully semi-prime when every proper F-submodule of $\mathcal{X}$ is semi-prime.

Definition 1.15 [14]:

For an F-module $\mathcal{A}$ in M, we associate the classical submodule $\mathcal{A}_* = \{x \in \mathrm{M}: \mathcal{A}(x) = 1 = \mathcal{A}(0_\mathrm{M})\}$.

2. PW-Closed Fuzzy Submodules

The preceding notions provide the setting for the proposed separation condition. The next definition recalls the classical pseudo w-closed notion and then gives its fuzzy counterpart.

Definition 2.1 [15]:

A submodule N of an R-module M is pseudo w-closed (briefly, pw-closed) in M if, for every m $\in \mathrm{M}$, m $\notin \mathrm{N}$ there exists a w-closed submodule K of M such that $\mathrm{N} \subseteq \mathrm{K}$ and m $\notin \mathrm{K}$.

Definition 2.2:

A F-submodule A of an F-module X over an R-module M. is called pseudo w-closed F-submodule(denoted by pw-closed), if for every F-point $x_t \subseteq \mathrm{X}$ with $x_t \nsubseteq A$, there exists a w-closed F-submodule B of $\mathrm{X} \subseteq \mathrm{A}$ and $x_t \nsubseteq B$.

Whenever the correspondence with associated classical submodules is used, condition (*) is assumed :

$A_* \subseteq B_*$ iff $\mathrm{A} \subseteq \mathrm{B}$. This condition provides the order compatibility needed to transfer pw-closedness between the fuzzy module and its associated classical module.

Proposition 2.3:

Assume that condition (*). For X be a F-module over an R-module M and an F-submodule A of X . Then A is a pw-closed in X if and only if A_* is a pw-closed of X_*.

Proof:

Suppose first that A is pw-closed in X. For $\mathrm{N} \leq X_*$ with x $\in X_*$, x $\notin \mathrm{N}$. define the characteristic F-submodule A: M $\longrightarrow [0,1]$, by:

$$\mathrm{A}\ (\mathrm{x}) = \begin{cases} 1 & if\ x \in N \\ 0 & otherwise \end{cases}.$$

Satisfying $\mathrm{A}_* = N$. The is a pw-closedness of A provides a w-closed F-submodule B with $A \subseteq \mathrm{B}$ and $x_1 \nsubseteq B$, by [5, proposition (2.4.3)],we obtain that B_* is a w-closed in X_*. Condition(*), given $\mathrm{N} = \mathrm{A}_* \subseteq B_*$, x $\notin B_*$. Hence A_* is a pw-closed in X_*.

Conversely, assume that A_* is a pw-closed submodule in X_* . For $x_t \subseteq X$ with $x_t \nsubseteq A$, we have x $\notin A_*$. By pw-closedness, of A_*, there exists w-closed B_* of X_* such that $A_* \subseteq B_*$ and x $\notin B_*$. By [5, Proposition 2.4.3], B is a w-closed F-submodule in X and condition (*), yields $\mathrm{A} \subseteq \mathrm{B}$, while $x_t \nsubseteq B$. Thus B satisfies the defining condition of pw-closedness, and A is pw-closed in X.

Remarks and Examples 2.4:

1. Every w-closed F-submodule of an F-module is pw-closed. Consequently, the class of w-closed F-submodules is contained in the class of pw-closed F-submodules. The inclusion is proper, as shown below.

Example:

Let M = $Z \oplus Z_2$ be the usual Z-module and consider the F-module X: M $\longrightarrow [0,1]$,

X (a, b) = 1, $\forall\ (a,b) \in Z \oplus Z_2$.

Define a F-submodule A : M $\longrightarrow [0,1]$, by: A (a, b) = $\begin{cases} 1 & if\ (a,b) \in (2,\overline{0}) \\ 0 & otherwise \end{cases}$

Since A_* = $< (2,\overline{0}) >$ is not w-closed submodules in X_* = M [15, Remark and Examples(3.1.2(1))], it follows from[5, Proposition (2.4.3)],that *A* is not w-closed F-submodules in X. On other hand, let B be any F-submodule of X satisfying A ≤ B ≤ X, there exists B : M ⟶ [0,1], with B (a, b) = $\begin{cases} 1 & if\ (a,b) \in Z \oplus \overline{(0)} \\ 0 & otherwise \end{cases}$

So that B_* = $Z \oplus \overline{(0)}$ is w-closed. By [15, Remark and Examples(3.1.2(1))], A_* is a pw-closed submodule in X_*= M. Proposition (2.3), yields that A is a pw-closed F-submodule of X. Hence the reverse implication fails. .

2. A direct summand of F-module need not pw-closed F-submodule.

Example:

Regard M = Z_{60} as Z-module in the usual manner and consider the F-module X: M ⟶ [0,1] define by: X (a) = 1, ∀ $a \in Z_{60}$. The F-submodules A and B : M → [0,1], are given: A (x) = $\begin{cases} 1 & if\ x \in (10) \\ 0 & otherwise \end{cases}$, and B (x) $\begin{cases} 1 & if\ x \in (\bar{6}) \\ 0 & otherwise \end{cases}$

Then A ⊕ B = X, so B is a direct summand of X. Nevertheless B is not pw-closed F-submodule. Indeed, define *C*: M → [0,1], by *C*(x) = $\begin{cases} 1 & \text{if } x \in (\bar{2}) \\ 0 & \text{otherwise} \end{cases}$, C_*= $(\bar{2})$

Then B_* = $(\bar{6}) \subseteq C_*$= $(\bar{2})$, while $(\bar{4}) \in C_*$, and $\overline{(4)} \notin B_*$. Hence $(\bar{4}) \subseteq (\bar{2})$, but $\overline{(4)} \notin (\bar{6})$. Therefore, the criterion established in proposition (2.3) is not satisfied. It follows that B is not a pw-closed F-submodule of X .

3. Unlike several closure properties, pw-closedness need not be invariant under isomorphic F-submodules. In particular, an F-submodule A that is a pw-closed in X may have an isomorphic counterpart B that is not be pw-closed in X. The following example demonstrates this behavior.

Example:

Take M = Z with its natural Z-module structure, and define the F-module X: M ⟶ [0,1] by: X (a) = 1, ∀ $a \in M$.

Define the F-subset A : M → [0,1], by A (x) = $\begin{cases} 1 & if\ x \in 2Z \\ 0 & otherwise \end{cases}$

It is straightforward to verify that A is an F-submodule of X and that A ≅ X . However, A is not a pw-closed F-submodule of X, since there exists no w-closed F-submodule satisfying the defining condition of pw-closedness. On the other hand, the whole F-module X is pw-closed because X is itself a w-closed F-submodule of X (see [5, Remark and Examples (2.4.4)(6)]. Therefore, although A is an F-submodule of the pw-closed F-submodule X, it fails to be pw-closed. This example confirms that the pw-closed property is not inherited by arbitrary F-submodules.

4. The class of pw-closed F-submodules is not hereditary with respect to inclusion. More precisely, for F-submodules A, B of X with A ⊆ B, pw-closedness of B in X need not be inherited.

Example:

Consider the F-module X : M ⟶ [0,1], where M = Z is a Z-module given by :

X (a) = 1, ∀ $a \in Z$.

with the F-submodule A : M ⟶ [0,1], define by: A (x) = $\begin{cases} 1 & if\ x \in 3Z \\ 0 & otherwise \end{cases}$

By part (3), X is pw-closed, whereas A is not pw-closed F-submodule in X because there is no a w-closed F-submodule of X contained separates every F-point outside A . Therefore A ⊆ X and X pw-closed do not imply that A is pw-closed.

5. Levelwise pw-closedness implies fuzzy pw-closedness.

Suppose that X is an a F-module over an R-module M, with A ≤ X. If each level submodule A_t is pw-closed in X_t for t ∈ [0,1], then A is a pw-closed in X.

Proof:

Let $x_t \subseteq \chi$ with $x_t \nsubseteq A$,. Since A_t is a pw-closed in X_t, $\forall$ t ∈ [0,1], it follows that, there exists a w-closed submodule B_t of X_t satisfying $A_t \subseteq B_t$ and x ∉ B_t, by [5, Remark and Examples (2.4.4)(8)], B is a w-closed in X such that $x_t \nsubseteq B$. Hence for every F-point $x_t \nsubseteq A$, one can find a w-closed F-submodule B of X containing A while excluding x_t. Therefore, A is a pw-closed of X.

Remark 2.5:

i- The class of pw-closed F-submodules is not closed under finite intersections. In particular, the intersection of two pw-closed F-submodules of a F-module need not be pw-closed. The following example illustrates this fact.

Example:

Throughout this example, let M = $Z_6 \oplus Z_2$ be a Z-module. Define X : M⟶ $[0,1]$,

X (a, b) = 1, $\forall\ (a,b) \in Z_6 \oplus Z_2$.

Consider the F-submodules A, B : M⟶ $[0,1]$, given by:

A(a, b) = $\begin{cases} 1 & if\ (a,b) \in <(\bar{0},\bar{1})> \\ 0 & otherwise \end{cases}$, B (a, b) = $\begin{cases} 1 & if\ (a,b) \in <(\bar{3},\bar{1})> \\ 0 & otherwise \end{cases}$

Then A_t = $<(\bar{0},\bar{1})>$ and B_t = $<(\bar{3},\bar{1})>$, $\forall$ t ∈ [0,1], are pw-closed submodules in X_t = M (See[15, Remark(3.1.2)]), so by Remarks and Examples (2.4)(5), it follows that both A and B belong to the class of pw-closed F-submodules in X. Since $(A \cap B)$(a,b) = min{(A (a, b), B (a, b)}, and consequently $(A \cap B)_t$ = {(0,0)}. The latter is not a pw-closed submodule of X_t [15, Remarks and Examples (3.2.1)(7)], Remark (2.4)(5) yields $A \cap B$ is not a pw-closed F-submodule of X.

This example shows that the intersection of two pw-closed F-submodule need not be pw-closed.

ii- The classes of closed F-submodules and pw-closed F-submodules are independent. Indeed, neither class is contained in the other, as illustrated by the following examples.

Example:

Let M = Z_{48} as Z-module. Define the F-module X: M ⟶[0,1] by X (α) = 1, $\forall\ \alpha \in \mathrm{Z}_{48}$. And let A :M ⟶ [0,1], by the following F-submodule :

A (x) = $\begin{cases} 1 & if\ a \in \langle\bar{3}\rangle \\ 0 & otherwise \end{cases}$

by [2, Remark and Examples (3.2) (2)], A is closed in X although it does not satisfy the w-closedness condition. Observe consider the F-submodule B:M ⟶ [0,1], given by:

B (x) = $\begin{cases} 1 & if\ a \in \langle\overline{16}\rangle \\ 0 & otherwise \end{cases}$

Consequently, B is pw-closed in X, while $\mathrm{A} \nsubseteq B$, hence a closedness alone does not guarantee pw-closedness.

Conversely, let M = Z_{32} Equip with the usual Z-module structure and let X: M ⟶[0,1], given by :

X (α) = 1, $\forall\ \alpha \in \mathrm{Z}_{32}$, with the F-submodule, A :M ⟶ [0,1], defined by :

A (x) = $\begin{cases} 1 & if\ a \in \langle\bar{2}\rangle \\ 0 & otherwise \end{cases}$

Since $A_t = \langle\bar{2}\rangle$, is a w-closed submodule in X_t, while A_t but not closed in X_t, it follows from [5, Remark and Examples (2..4.3)(6)], given that A is a w-closed F-submodule and hence a pw-closed F-submodule. On the other hand, A is not closed by [2, proposition (3.2)]. Therefore, a pw-closed F-submodule need not closed.

The pw-closed condition leads to several useful consequences.

Proposition 2.6:

Let A and B F-submodules of X such that $A \subseteq B$. Suppose that A is a pw-closed F in B

whereas B is a w-closed in X. Moreover, every weak essential F-submodule of A contains B. It follows that A is a pw-closed in X.

Proof:

For $x_t \subseteq X$ with $x_t \nsubseteq A$, the pw-closedness of A in B, gives a w-closed F-submodule C of B satisfying $A \subseteq C$ and $x_t \nsubseteq C$. Since B is a w-closed in X , [5, proposition (2.4.5)], gives that C is a w-closed in X. Consequently $A \subseteq C$ and $x_t \nsubseteq C$. Since x_t was chosen arbitrarily, it follows that for every F-point $x_t \nsubseteq A$ there exists a w-closed F-submodule containing A. this establishes the pwclosedness of A is a pw-closed in X.

We next present a sufficient condition ensuring that every closed F-submodule is pw-closed. The required notion is recalled below.

Definition 2.7 [5]:

An F-module X in the R-module M is completely essential when every non-trivial weak essential F-submodule of X is an essential in X.

Proposition 2.8:

A non-trivial closed F-submodule A of X is pw-closed in X, whenever all weak essential F-submodule of X are completely essential.

Proof:

By [5, proposition (2.4.9)], the given assumptions imply that every closed F-submodule of X is w-closed. In particular, A is a w-closed in X. The assertion now follows Remark (2.4) (1), which places A among the pw-closed F-submodule of X.

Proposition 2.9:

Every non-trivial closed F-submodule a fully semi-prime F-module is pw-closed.

Proof:

The fully semi-prime property, together with [5, Proposition (2.4.10)], implies that every non-trivial closed F-submodule is w-closed. Remark 2.4(1) then yields pw-closedness.

Proposition 2.10:

If B contains every w-closed F-submodule of X and A is pw-closed in X, then A is pw-closed in B.

Proof:

For F-point $x_t \subseteq B$ with $x_t \nsubseteq A$, there exists a w-closed F-submodule C of X such that $A \subseteq C$, and $x_t \nsubseteq C$. By hypothesis $C \subseteq B$, so C witnesses the pw-closedness of A in B.

The following result characterizes the behavior of pw-closed F-submodules under direct sums, provided that the underlying modules satisfy the required essentiality assumptions.

Proposition 2.11:

Suppose X and Y are F-modules over R-modules M_1, M_2 (respectively), with $\mathrm{ann}X_* + \mathrm{ann}Y_* = R$ and every weak essential submodule is completely essential, if $A \subseteq X$ and $B \subseteq Y$ are non-trivial pw-closed F-submodules, then $A \oplus B$ is pw-closed in $X \oplus Y$.

Proof:

Proposition (2.3), given that A_* and B_* are pw-closed submodules of $X_*$and Y_* (respectively). By [15, proposition (3.1.9)] $A_* \oplus B_*$ is a pw-closed submodule in $X_* \oplus Y_*$. Since $(A \oplus B)_* = A_* \oplus B_*$ by[1, lemma (2.3.3)], it follows that $(A \oplus B)_*$ is a pw-closed submodule of$(X \oplus Y)_*$. Invoking proposition (2.3), once again, we conclude that $A \oplus B$ is is a pw-closed F-submodule of $X \oplus Y$.

Proposition 2.12:

Under the assumptions of Proposition (2.11), The direct sum $A \oplus B$ is a pw-closed is pw-closed in $X \oplus Y$ iff A and B are pw-closed F-submodule in X and Y (respectively).

.

Proof:

(⇒) If $A \oplus B$ is a pw-closed F-submodule of $X \oplus Y$. First, proposition (2.3), given that $A_* \oplus B_*$ is a pw-closed in $X_* \oplus Y_*$. Next, by [1, lemma (2.3.3)] $(A \oplus B)_* = A_* \oplus B_*$. consequently,[15, proposition (3.1.10)] implies that A_* and B_* are pw-closed submodules of X_* and Y_*(respectively). Finally another application of proposition (2.3) yields that A and B are pw-closed in X and Y (respectively),

(⇐) The is proposition (2.11).

Corollary 2.13:

Assume that X, Y are an F-modules over an R-module M_1, M_2 (respectively), satisfying annX_* +ann$Y_* = R$ and that every submodule of $X_*$and Y_* fully semi-prime. Then $A \oplus B$ is a pw-closed F-submodule of $X \oplus Y$ iff corresponding F-submodules A of X and B of Y are both pw-closed in X and Y (respectively).

Proof:

Every fully semi-prime F-submodule is completely essential, under the hypotheses of used in [5], so Proposition (2.12), applies.

Proposition 2.14:

. Suppose X satisfies condition (*) and every weak essential F-submodule is completely essential. If A is pw-closed in B and B is pw-closed in X, then A is pw-closed in X.

Proof:

The argument follows Proposition (2.6), with [5, Proposition (2.4.12)] supplying the corresponding classical result.

Proposition 2.15:

If X is fully semi-prime, A ⊆ B ⊆ X, A is pw-closed in B, and B is pw-closed in X, then A is pw-closed in X.

Proof:

By [5, Proposition (2.4.13)], every F-submodule of a fully semi-prime F-module is completely essential. Proposition (2.14), therefore applies.

For use in the subsequent result, recall from [5], that an F-submodule A of X is y-closed whenever X/A is a non-singular F-module.

Proposition 2.16:

Every non-trivial y-closed F-submodule of a fully semiprime F-module X over M, where M is an R-module, is pw-closed.

Proof.

Let a non-trivial y-closed F-submodule A of X. According to [5, Proposition (2.5.15)], A is a w-closed in X. The implication from w-closedness to pw-closedness follows from Remark and Examples (2.4) (1)). Hence A is pw-closed in X .

Proposition 2.17:

Suppose the relevant level module X_t is fully semiprime. if A is pw-closed F-submodule in X and B is weak essential in X, then $A \cap B$ is a pw-closed F-submodule in B.

Proof:

Proposition (2.3), gives A_* is a pw-closed in X_*. By [5, Proposition (2.3.8)], B_* is weak essential in X_*. Hence [15, Proposition (3.1.15)], applies to obtain that $A_* \cap B_*$ is pw-closed in B_*. By Remark (1.5)(2) $(A \cap B)_* = A_* \cap B_*$. Consequently, $(A \cap B)_*$ is a pw-closed in B_*. Another application Proposition (2.3), conclude that $A \cap B$ is a pw-closed F-submodule in B.

Corollary 2.18:

If X is fully semi-prime and A, $B \leq$ X with B is weak essential in X, then A ∩ B is pw-closed in B provided that A is either w-closed or y-closed in X .

Proof.

By [5, Proposition (2.3.21)], the corresponding level module is fully semi-prime. Proposition (2.17), therefore gives both assertions

The next result shows that pw-closedness is preserved under surjective F-homomorphic images.

Proposition 2.19:

Let $f : \mathrm{X} \longrightarrow Y$, be an F-epimorphism with X and Y F-modules over the R-modules M_1,M_2 respectively. Assume that A is a pw-closed in and that f-invariance holds for A and every w-closed F-module of X containing A . Then f (A) is a pw-closed in Y.

Proof:

Take an F-point $y_t \subseteq Y$ such that $y_t \nsubseteq f(A)$. The suriectivity f, guarantees there existence $x_t \subseteq X$ with f(x_t) = y_t. Clearly $x_t \nsubseteq A$, since $x_t \subseteq A$ would lead to y_t= f(x_t) $\subseteq$ f(A), contradicting the choice of y_t . As A is a pw-closed F-submodule of X, there is a w-closed F-submodule satisfying $A \subseteq B$ and $x_t \nsubseteq B$, the f-invariance of B together with [5, proposition (2.4.17)],implies that f (B) is a w-closed F-submodule of Y. Furthermore from $A \subseteq B$, we obtain f(A) $\subseteq$ f(B). suppose that $y_t \subseteq f(B)$. Since f(x_t) = y_t, it follows that $x_t \subseteq f^{-1}(f(B))$ using the f-invariance of B , we have $f^{-1}(f(B))$ = B and hence $x_t \subseteq B$, which contradicts the construction of B . Therefore, $y_t \nsubseteq$ f(B). Thus, f(B) separates every F-point outside f (A) from f (A). Hence, by the definition of pw-closed F-submodules f(A) is a pw-closed F-submodule of Y.

3. PW-Extending Fuzzy Modules

The structural properties of pw-closed F-submodules established above motivate the study of F-modules in which pw-closed submodules split. Recall that an F-module X is w-extending if every w-closed F-submodule is contained in a direct summand of X [5]. In the classical setting, pw-extending modules are defined through containment of each pw-closed submodule in a direct summand [15]. In the present paper we use the stronger direct-summand formulation, which is particularly adapted to the decomposition results below.

Definition 3.1:

The F-module X associated with an R-module M, is said to be pw-extending whenever each pw-closed F-submodule of occurs as a direct summand of X.

Remark and Examples 3.2:

1. Every pw-extending F-module is a w-extending F-module, but the converse does not hold.

Proof:

Assume that X is pw-extending F-module and let A be a w-closed F-submodule of X. By the inclusion relationship between the two closure classes, every w-closed F-submodule is also pw-closed by Remark (2.4) (1). Since X is assumed to be pw-extending, it follows directly from the definition A is a direct summand of X. Consequently, X satisfied the defining property of a w-extending F-module establishing the desired implication.

Example:

Consider the Z-module M = Z_{16} and define the associated F-module X: $\mathrm{M} \longrightarrow$[0,1]

X (α) = 1, $\forall$ $\alpha \in \mathrm{Z}_{16}$

Then $X_* = \mathrm{Z}_{16}$ is w-extending module [15, Remark and Examples (4.2.2)(1)] and the level characterization in [5, Proposition

(3.2.2)] gives that X is w-extending. To verify that the converse implication fails, consider the F-submodule A:M ⟶ [0,1],

A (x) = $\begin{cases} 1 & if\ a\ \in \langle \bar{2} \rangle \\ 0 & otherwise \end{cases}$

The corresponding level submodule is A_*= $\langle \bar{2} \rangle$. According to [15, Remark and Examples (4.2.2)(1)], A_* is a pw-closed submodule of X_*. By the level characterization in Proposition (2.3), this implies that A is a pw-closed F-submodule of X. Furthermore by [15, Remark and Examples (4.2.2)(1)] states that A_* is not a direct summand of X_*. Applying of [5, Lemma (2.1.21)], we conclude that A is not a direct summand of X . Hence, X contains a pw-closed F-submodule which is not a direct summand, and so X is not a pw-extending F-module. Consequently, X is a w-extending F-module that is not pw-extending. This example demonstrates that the converse implication does not hold in general.

2. Any semi-simple F-module is pw-extending.

Proof:

Let X is a semi-simple. For any F-submodule A of X, there exists an F-submodule C of X such that X = A⊕C. Thus every pw-closed F-submodule A of X is a direct summand of X and consequently X is pw-extending.

The converse is not valid in general, as demonstrated by the following example.

Example:

The Z-module M = $Z_6 \oplus Z_2$ is equipped the F-module X : M⟶ [0,1], where :

X (a, b) = 1, ∀ $(a,b) \in Z_6 \oplus Z_2$.

The pw-closed F-submodules of X are precisely A, B and X, where A, B: M⟶ [0,1], : A (a, b) = $\begin{cases} 1 & if\ (a,b) \in < (\bar{0},\bar{1}) > \\ 0 & otherwise \end{cases}$, B (a, b) = $\begin{cases} 1 & if\ (a,b) \in < (\bar{3},\bar{1}) > \\ 0 & otherwise \end{cases}$

Each of A, B and X is a direct summand of X, hence X is a pw-extending F-module. Nevertheless X is not semi-simple, since some F-submodules of X are not direct summand. Thus the converse fails.

3. The notions of pw-extending F-modules and extending F-modules are independent. The following example establishes that the extending property does not imply the pw-extending property. Following [5], an F-module X over an R-module M we say that X has the extending property (briefly, it is an F-CS module) if for every F-submodule A of X there exists a direct summand B of X such that A $\leq_e$ X.

Example:

Over the Z-module M = Z_{16}, the F-module X : M ⟶[0,1], satisfies

X (α) = 1, ∀ $\alpha \in Z_{16}$.

Since every F-submodule of X is essential in X itself, and X is a direct summand of X, the extending condition is satisfied. Hence X is an extending. On the other hand, X is not a pw-extending F-module, because it contains pw-closed F-submodule of X that is not a direct summand of X. Hence extendingness does not imply pw-extendingness.

Conversely, take M = $Z_6 \oplus Z_2$ with its Z-module structure and define the F-module X by X : M⟶ [0,1], X (a, b) = 1, ∀ $(a,b) \in Z_6 \oplus Z_2$. Then X is pw-extending. To show that X is not extending consider the F-submodule A :M ⟶ [0,1] A (x) = $\begin{cases} 1 & if\ a\ \in \langle (\bar{2},\bar{1},) \rangle \\ 0 & otherwise \end{cases}$

The associated classical submodule A_*= $\langle (\bar{2},\bar{1},) \rangle$, has no proper essential submodule.

Hence A_* is closed in X_*. By [5, proposition (3.2)], A is closed in X. Moreover, A_* is not direct summand of X_*. It follows [5, lemma (2.2.21)] that A is not direct summand of X. Therefore, A does not satisfy the condition required for an extending F-module and X is not extending. Thus X is pw-extending but not extending.

The pw-extending property passes between X and X_*.

Theorem 3.3:

Let X is an F-module satisfying condition (*). The pw-extending property of X is equivalent to that of its associated module X_* .

Proof

Suppose first that X is a pw-extending F-module. Let N is pw-closed submodule of X_* and A denoteF-submodule of X corresponding to N given by:

$$\mathrm{A}\ (x) = \begin{cases} 1 & if\ x \in N \\ 0 & otherwise \end{cases}.$$

By the construction of A, its associated submodule satisfies $\mathrm{A}_* = N$. Proposition (2.3) establishes that A is a pw-closed in X. The pw-extending property of X , therefore yields that A is splits as a direct summand of X. By [5, Lemma (2.1.21)], together with condition (*). The corresponding submodule $\mathrm{A}_* = N$ is a direct summand of X_*. Hence every pw-closed submodule of X_* direct summand and X_* is pw-extending module.

Conversely, assume that X_* is a pw-extending module and let A be a pw-closed F-submodule of X. By Proposition (2.3), the associated submodule A_* is pw-closed in X_*. Consequently A_* a direct summand of X_*. Applying [5, Lemma (2.1.21)], under condition (*), this direct summand decomposition corresponds to decomposition of X, in which A is a direct summand. Thus every pw-closed F-submodule of X is a direct summand of X and therefore X is a pw-extending.

Proposition 3.4:

If X is fully semi-prime and pw-extending F-module, then every direct summand of X is pw-extending F-module.

Proof:

Assume that A be a direct summand of X, so X = $A \oplus B$ for some F-submodule B of X. Take a pw-closed F-submodule *C* of A. Since A is a direct summand of X, it is a closed F-submodule of X (See [5, Remark and Examples (2.1.15) (2)]). Hence, by proposition (2.9) and the fully semi-prime assumption on X, *C* is also pw-closed F-submodule of X. As X is a pw-extending, there exists an F-submodule D of X such that X = $C \oplus$D. Because $C \leq A$, restricting decomposition to A yields A = $C \oplus$(D $\cap A$), where the sum of is direct, as $C \cap D = 0_1$. Thus *C* is a direct summand of A. Therefore, every pw-closed F-submodule of A is a direct summand. Consequently, A is a pw-extending F-module.

Proposition 3.5:

In a pw-extending F-module X, the intersection of any two F-submodules A and B is a direct summand of both A and B, whenever $A \cap B$ is pw-closed.

Proof:

The pw-extending property of X guarantees the existence of an F-submodule *C* of X satisfying X = (A $\cap B$)$\oplus$*C*. intersecting this decomposition with A given A = A $\cap X$ = $[(A \cap B) \oplus C] \cap A$, using [5, Lemma (3.1.4)], we obtain A = $(A \cap B) \oplus (C \cap A)$, so $A \cap B$ is a direct summand of A. Using the same decomposition and intersecting it with , B we get $B = B \cap X = [(A \cap B) \oplus C] \cap B$, again by [5, Lemma (3.1.4)]. Hence $A \cap B$ is a direct summand of B Hence $A \cap B$ is a direct summand of both A and . B

The following criterion identifies a setting in which the notion of a pw-extending F-module agrees with the classical concept of an extending F-module.

Proposition3.6:

If every weak essential extension of an F-submodule of X is completely essential, then every pw-extending F-module X is extending.

Proof.

Take a non-empty closed F-submodule A of X. By Proposition (2.8), under the stated assumption, implies that A is pw-closed in X. The pw-extending property of X, then ensures that A is a direct summand of X. As this holds for every non-empty closed F-submodule of X, the defining condition of an extending F-module is fulfilled.

Proposition 3.7:

Every fully semi-prime pw-extending F-module is an extending F-module.

Proof:

By proposition (2.9), closed F-submodules of a fully semi-prime F-module are necessarily pw-closed. Appling the pw-extending property of X shows that each closed F-submodule is a direct summand of X. Therefore, X is an extending F-module.

Proposition 3.8:

Suppose that X is pw-extending and that

each F-submodule of X is essential in pw-closed F-submodule of X. Then X is extending.

Proof:

Choose any F-submodule A of X. By the assumed condition,, there is a pw-closed F-submodule B of X in which A is essential. Since X is pw-extending, B is a direct summand of X. Therefore, A is essential in a direct summand of X . As A was arbitrary, X satisfies the extending property.

The following proposition places the pw-extending property within the framework of two well-known classes of F-modules namely FI-extending and CLS F-modules. For convenience, the corresponding definitions are recalled below.

Definition 3.9 [5]:

An F-module X is called FI-extending, if every fully invariant is essential in a direct summand of X.

Definition 3.10 [5]:

Let M be an R-module and X an F-module over M. the module X is said to be CLS-(F-module) whenever each y-closed F-submodule of X is a direct summand of X .

Proposition 3.11:

Every fully semi-prime pw-extending module satisfies each the following properties .

1. X is an FI-extending F-module.
2. X is CLS(F-module).

Proof:

1. Proposition (3.7), shows that every fully semi-prime pw-extending F-module is extending. By [16, Remark (3.11)], states that every extending F-module is FI-extending F-module. Combining these two facts yields the desired conclusion.

2. By Proposition (2.16) identifies every y-closed F-submodule of fully semi-prime F-module as pw-closed. Since X is pw-extending each such F-submodule necessarily a direct summand of X. Consequently, every y-closed F-submodule is a direct summand. Therefore, X is a CLS (F-module).

Open Problems

1. Develop a fuzzy analogue of quasi-closed submodules and determine its characterizations, closure behavior, and

relationships with established classes of fuzzy submodules.

2. Develop a fuzzy version of fully extending modules and investigate its structural characterizations, permanence properties, and interaction with pw-extending and extending F-modules.

4. Conclusion

The study establishes a structural framework for pw-closed F-submodules and the corresponding pw-extending fuzzy modules. The results show that pw-closedness properly enlarges the w-closed class, while the examples identify failures of heredity and finite-intersection closure and separate pw-closedness from ordinary closedness. Condition (*) gives the required correspondence with associated classical submodules and permits classical structural results to be transferred to the fuzzy setting. The pw-extending theory then clarifies the role of direct summands and identifies hypotheses under which pw-extending modules become extending. In particular, full semi-primality yields inheritance by direct summands and leads to FI-extending and CLS consequences. These results provide a basis for further study of generalized closedness and extending phenomena in fuzzy algebra.

The pw-extending property is then placed within the broader theory of w-extending and extending F-modules. Direct-summand inheritance is obtained under full semi-primality, and several criteria are established for pw-extending modules to be extending. In addition, fully semi-prime pw-extending F-modules are shown to be FI-extending and CLS. These results clarify the structural role of pw-closedness and provide a basis for further development of generalized extending properties in fuzzy algebra.

Reference

1. Jari, R. H. "Prime Fuzzy Submodules and Prime Fuzzy Modules", M. Sc. Thesis, University of Baghdad, (2001)
2. H. K. Marhon and H. Y. Khalaf, "Essential Fuzzy Submodules and Closed Fuzzy submodules". Iraq . J. of Science, Vol. 61, No. 4, pp. 890-897, (2020).
3. Hadi, I. M. A, "Semi-Prime Fuzzy submodules of Fuzzy modules", Ibn Al-Haitham J., for Pure and Appl. Sci., 17(3): 112-123, (2004).
4. H. K. Marhon and H. Y. Khalaf, "Weak essential Fuzzy Submodules of Fuzzy Modules" Ibn Al-Haitham J., for Pure and Appl. Sci., 33(4) (2020).
5. Marhoon, H. K." Fuzzy Closed Submodules and Fuzzy W-closed Submodules with the some of Thier Generalization ", Ph. D Thesis, University of Baghdad, . (2020)"
6. Zadeh, L. A. " Fuzzy Sets, Information and Control", 8, 338-353, (1965)
7. Zahedi, M. M. "On L-Fuzzy Residual Quotient Module and P. Primary Submodule", Fuzzy Sets and Systems, 51,333-344, .(1992).
8. Martinez, L. "Fuzzy Module Over Fuzzy Rings in Connection with Fuzzy Ideals of Rings",J. Fuzzy Math.,4,843-857, (1996)
9. Kumar, R "Fuzzy Semi-primary Ideals of Rings", F-sets and Systems, Vol.42, pp.263-272 ,(1991)
10. Kumar R., S. K. Bhambir, Kumar P. "Fuzzy Submodule of Some Analogous and Deviation", Fuzzy Sets and Systems, Vol.70, PP.125-130 ,(1995).

11. H. G. Rashed, "Fully Cancellation Fuzzy Modules and Some Generalizations", M. Sc. Thesis, University of Baghdad, (2017).
12. A. H., Maysoun,"F-regular Fuzzy Modules", M. Sc. Thesis, Univ. of Baghdad, (2002).
13. J. A. Khubchandani, P.A. Khubchandani, " A study of Fuzzy W-closed Submodules And Related Concepts" TWMS J. App. And Eng. Math. Vol. 16, N. 4, pp. 472-481, (2026).
14. Mashinchi, M. and Zahedi, M. M, "On L-Fuzzy Primary Submodule", Fuzzy Sets and Systems, Vol. 49, pp. 231-236, (1992).
15. Mohammed, E. D.,"W-closed Submodule and W-extending Modules with Some Generaliztions", M. Sc. Thesis, Univ. of Tikrit, (2018).
16. H. K. Marhon " S-extending Fuzzy Modules ", Center Asian Journal of Math. Theory and Computer Scince 6(3), (2025).