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On Julia and Mandelbrot sets of gamma and beta functions for fractal images

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ABSTRACT

This work explores the Julia and Mandelbrot sets of the gamma and beta functions by using Lanczos approximation procedures for fractal images. Various Julia and Mandelbrot sets associated with the gamma function are generated using the iterative function, $f_\lambda(z)$, with varoius λ-values. The Lanczos approximation of the gamma function presents an efficient and easy algorithm to compute an approximate solution of the iterative gamma function to an arbitrary precision. The resulting images reveal that the fractal (chaotic) behavior found in elementary functions is also found in the gamma functions. The chaotic nature of the Julia and Mandelbrot sets provides an understanding of complexity in systems as well as in shapes. The relationship between the gamma and beta functions was exploited to obtain the Lanczos approximation for beta function.

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1. Introduction

Dynamical systems are a well-known branch of mathematics, but until the arrival of computers the sheer number of calculations involved makes it impractical for real use. The computer's ability to perform rapid calculations allows us to condense billions of calculations into results we can digest. Benoit Mandelbrot is the first person to use a computer program to produce graphical representations of dynamical systems in the complex plane, based on the quadratic formulas described by Amobeda et al. (2016). The Julia and Mandelbrot sets have been studied extensively over the years using elementary polynomials functions. In recent times, the study has been extended beyond the elementary functions to a special function called the Riemann zeta function (see Woon, 1998). This extension to higher function was the first of its kind; there are many other special functions in mathematics. Mandelbrot created the first-ever "theory of roughness" and he saw roughness in the shapes of mountains, coastlines, river basins, the structure of plants, blood vessels, lungs, and the clustering of galaxies (Mukherjee, 2017). This study is borne out of the desire to know how these sets behave with other special functions, in particular gamma and beta functions.

Abbas et al. (2020) discussed the generation of Julia and Mandelbrot sets via fixed points. Using the Picard–Mann iteration, Cui et al. (2020) introduced the concept of Julia and Mandelbrot sets. The dynamics of Julia and Mandelbrot sets over hyperbolic numbers are given in Mark's (2004) study. Htun's (2019) analysis of three polynomial sets, i.e., Newton's method, Julia and

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Mandelbrot sets, and the comparison of two fractals, is carried out using MATLAB. In Amobeda et al.'s (2016) study, the Julia and Mandelbrot sets of gamma function using Lanczos approximation were obtained.

In Nazeer et al.'s (2015) study, the fixed point result in generation of Julia and Mandelbrot sets using Jungck Mann and Jungck Ishikawa iteration method is discussed. Panwar (2014) presented the complex dynamics of new-fangled fractal controlled by transcendental function (i.e., Tan function) using iterative procedure known as Ishikawa iteration method. Panwar and Mishra (2014) used the Ishikawa iteration method and generated a Julia set using Cos function and produced beautiful images full of colorful spirals and other shapes. Kumari et al. (2016) obtained the generalization of new Julia and Mandelbrot sets for a new faster iterative process. Ankit et al. (2014) applied a 3D iterated function system in the generalization of fractal images for Mandelbrot and Julia sets. Panwar et al. (2013) presented and analyzed the fractal generated using the logarithmic function of Mandelbrot and Julia sets for integer and no-integer values.

The generation of new Julia sets and Mandelbrot sets for inverse sine function by applied Ishikawa iteration is discussed in Rajeshri and Yashwant's (2013) study.

Mohammad and Azman (2007) obtained a new cryptographic key exchange protocol. Our desire is to know how these sets behave with other special functions like the gamma and beta functions. The gamma function is only defined for real arguments; therefore, there is a need to study the Lanczos approximation of the gamma function of a complex argument. In this study, we use the relationship between gamma and beta functions to define the beta function of complex argument using the Lanczos approximation. This study presents Julia and Mandelbrot sets of gamma function for fractal images and Lanczos approximation for beta function.

1.1. Mathematical preliminaries

1.1.1. Julia set

Let $f_\lambda(z) = z^2 + \lambda, \quad z,\lambda \in \mathbb{C}$ (1)

Julia set is the set of all points whose nth iterate, f_λ^n, converges as n and tends to infinity. The *filled-in Julia set* of a polynomial, *f*, denoted by $\boldsymbol{B}_\lambda$, is the set of all points with points with bounded orbits under *f* mathematically

$$B_\lambda = \{z : \left|f_\lambda^n(z)\right| \neq \infty \text{ as } n \to \infty\}. \tag{2}$$

The *Julia set* of *f*, denoted by J_λ, is the boundary of the filled-in Julia set.

1.1.2. Mandelbrot set

Let $f_\lambda(z) = z^2 + \lambda, \lambda = a + ib$ be the iterate function. Mandelbrot set is made up of all the values of λ for which the corresponding orbit diverges to infinity as we iterate the function. Mathematically,

$$M = \{\lambda : \lambda \in \mathbb{C}, \lim_{n\to\infty} f_\lambda^n(z) \neq \infty \text{ as } n \to \infty\}. \tag{3}$$

1.1.3. Beta function

The beta function is defined in $\mathbb{C} \times \mathbb{C}$ as follows:

$$\beta(z_1, z_2) = \int_0^1 s^{z_1 - 1}(1 - s)^{z_2 - 1} ds \tag{4}$$

This integral often is called beta integral.

1.1.4. Gamma function

The gamma function, Γ (z), is a meromorphic function that arises frequently in a complex analysis. It is defined in the right half complex plane by

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds, \mathbb{R}(z) > 0 \tag{5}$$

This is a fundamental recurrent relationship for gamma function

$$\Gamma(n + 1) = n\Gamma(n) \tag{6}$$

2. Methodology

2.1. Lanczos gamma function approximation

In order to produce the graphic representation of Julia sets and Mandelbrot sets, we will need the Lanczos gamma approximation.

$$\Gamma(z+1) = \sqrt{2\pi}\left(z + r + \frac{1}{2}\right)^{z+\frac{1}{2}} e^{-(z+r+\frac{1}{2})} S_r(z) \tag{7}$$

2.2. Relationship between beta function and gamma function

The relationship between beta function and gamma function is given by

$$\beta(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)} \text{ where } \mathbb{R}(z_1, z_2) > 0.$$

2.3. Lanczos beta function approximation

From the relationship between the gamma and beta functions after the approximation, we have

$$\beta(z_1, z_2) = \frac{z_3\Gamma(z_1 + 1)\Gamma(z_2 + 1)}{z_1 z_2 \Gamma(z_3 + 1)} \tag{8}$$

Lanczos approximation for $\Gamma(z_1+1), \Gamma(z_2+1), \Gamma(z_3+1)$, is given by

$$\Gamma(z_n+1)=\sqrt{2\pi}\left(z_n+r+\frac{1}{2}\right)^{z_n+\frac{1}{2}} e^{-(z_n+r+\frac{1}{2})} S_r(z_n)$$

For n = 1, 2, and 3.

By using Equation (8)

$$=\frac{z_3}{z_1z_2}\cdot\frac{\sqrt{2\pi}\left(z_1+r+\frac{1}{2}\right)^{z_1+\frac{1}{2}} e^{-(z_1+r+\frac{1}{2})} S_r(z_1).\sqrt{2\pi}\left(z_2+r+\frac{1}{2}\right)^{z_2+\frac{1}{2}} e^{-(z_2+r+\frac{1}{2})} S_r(z_2)}{\sqrt{2\pi}\left(z_3+r+\frac{1}{2}\right)^{z_3+\frac{1}{2}} e^{-(z_3+r+\frac{1}{2})} S_r(z_3)} \tag{9}$$

where

$$S_r(z_n)=\frac{1}{2}a_0(r)+a_1(r)\frac{z_n}{z_n+1}+a_2(r)\frac{z_n(z_n-1)}{(z_n-1)(z_n-2)}+\ldots$$

for n = 1,2,3

Take $a_0(r)=\sqrt{\dfrac{2e}{\pi(r+\frac{1}{2})}.e^r}$, and

$$a_n(r)=\sum_{k=1}^{n} C(2n+1,2k+1)\frac{\sqrt{k}}{\pi}\left(k-\frac{1}{2}\right)!X\left(k+r+\frac{1}{2}\right)^{-\left(k+\frac{1}{2}\right)} e^{\left(k+r+\frac{1}{2}\right)}$$

For $n \geq 1$

Equation (9) give the required result of Lanczos beta function approximation.

With $C(i, j)$ denoting the (i, j)th element of the Chebyshev polynomial coefficient matrix which can be calculated from the identity

$C(1, 1) = 1$; $C(2, 2) = 1$

$C(i,1) = -C(i-2,1)$, for $i = 3, 4,\ldots$

$C(i, j) = 2C(i-1,j-1)$ for $i = j = 3, 4,\ldots$

$C(i, j) = 2C(i-1,j-1)-C(i-2,j)$ for $i > j = 2, 3,\ldots$

To derive Equation (7), we use Equations (10) and (11) which are Stirling's and Spouge's series

$$\Gamma(z+1) \approx \sqrt{2\pi z}\ (z^z e^{-z}) \text{ as } |z| \to \infty \tag{10}$$

$$\Gamma(z+1) = (z+a)^{z+\frac{1}{2}} e^{(z+a)} (2\pi)^{\frac{1}{2}}\left[c_0+\sum_{k=1}^{N}\frac{c_k}{z+k}+\varepsilon(z)\right] \tag{11}$$

The Lanczos approximation (Eq. 7) was arrived at using Euler's formula (Eq. 5) as the starting point and transforming the integral in a series of steps.

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt$$

Let $\alpha t \to t, \Rightarrow \alpha dt = dt$

$$\Gamma(z+1)=\int_0^\infty(\alpha t)^z e^{-\alpha t}\alpha dt=\alpha^{z+1}\int_0^\infty t^z e^{-\alpha t}dt \tag{12}$$

Let $\alpha = 1+pz,\ p>0$ substituting into Equation (12)

$$\Gamma(z+1)=(1+pz)^{z+1}\int_0^\infty t^z e^{-(1+pz)t}dt$$

$$\Gamma(z+1)=(1+pz)^{z+1}\int_0^\infty (te^{-pt})^z e^{-t}dt \tag{13}$$

Introducing $\left(\frac{ep}{ep}\right)^z$ into Equation (13), the evaluation yields

$$\Gamma(z+1)=\left(\frac{ep}{ep}\right)^z(1+pz)^{z+1}\int_0^\infty (te^{-pt})^z e^{-t}dt$$

Let $v=e^{1-pt}$, then $dv=-pe^{1-pt}dt \Rightarrow dt=\dfrac{dv}{-pv}$ and

$$t=\frac{1}{p}-\frac{1}{p}\log v$$

When $t\to 0, v\to e$ and $t\to\infty, v\to 0$

$$\Gamma(z+1)=(1+pz)^{z+1}(ep)^z\int_e^0[v(1-\log v)]^z e^{-(\frac{1}{p}-\frac{1}{p}\log v)}\frac{dv}{-pv}$$

Since $\int_e^o=-\int_0^e$

$$\Gamma(z+1)=\left[p\left(\frac{1}{p}+z\right)\right]^{z+1}(ep)^{-z}\int_0^e\left[v(1-\log v)\right]^z e^{-(\frac{1}{p}-\frac{1}{p}\log v)}\frac{dv}{pv}$$

$$\left(\frac{1}{p}+z\right)^{z+1}p^{z+1}(ep)^{-z}p^{-1}\int_0^e\left[v(1-\log v)\right]^z e^{-(\frac{1}{p}-\frac{1}{p}\log v)}\frac{dv}{v} \tag{14}$$

After simplifying Equation (14), the equation becomes

$$\Gamma(z+1)=\left(\frac{1}{p}+z\right)^{z+1}e^{-z}\int_0^e\left[v(1-\log v)\right]^z e^{-\frac{1}{p}}v^{\frac{1}{p}-1}dv$$

$$=\left(\frac{1}{p}+z\right)^{z+1}e^{-z-\frac{1}{p}}\int_0^e\left[v(1-\log v)\right]^z v^{\frac{1}{p}-1}dv \tag{15}$$

Let , so that

Equation (15) becomes

$$\Gamma(z+1)=(z+r+1)^{z+1}e^{-(z+r+1)}\int_0^e[v(1-\log v)]^z v^r dv \tag{16}$$

Let $\cos^2\theta=[v(1-\log v)]$, then $dv=\dfrac{2\sin\theta\cos\theta}{\log v}d\theta$, when $v=0, \theta=\dfrac{-\pi}{2}$ and $v=e, \theta=\dfrac{\pi}{2}$.

$$\Gamma(z+1)=(z+r+1)^{z+1}e^{-(z+r+1)}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(\cos^{2z}\theta)v^r\frac{2\sin\theta\cos\theta}{\log v}d\theta$$

(17)

Writing $2=\sqrt{2}.\sqrt{2}$ Equation (17) becomes

$$\Gamma(z+1)=\sqrt{2}(z+r+1)^{z+1}e^{-(z+r+1)}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\cos^{(2z+1)}\theta\,\frac{\sqrt{2}v^{r}\sin\theta}{\log v}\,d\theta$$

Let $z\to z-\dfrac{1}{2}$, then

$$\Gamma(z+\frac{1}{2})=p_r(z)\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(\cos^{2z}\theta)\left[\frac{\sqrt{2}\sin\theta\cos\theta}{\log v}\right]d\theta \tag{18}$$

where $p_r(z)=\sqrt{2}\left(z+r+\dfrac{1}{2}\right)^{z+\frac{1}{2}}e^{-(z+r+\frac{1}{2})}$

The integral is evaluated using Fourier series expansion to give the required result

$$\Gamma(z+1)=\sqrt{2\pi}\left(z+r+\frac{1}{2}\right)^{z+\frac{1}{2}}e^{-(z+r+\frac{1}{2})}S_r(z)$$

which is regarded as the Lanczos gamma function approximation for Julia and Mandelbrot sets.

3. Result

3.1. Julia sets for elementary function

Figures 1–4 show the Julia set for functions. The blue region of the plane is the filled-in Julia set which correspond to those points whose orbits are bounded. The boundary of the filled-in Julia set forms the Julia set. Most of the rest of the plane is colored red shading to some other colors. In this colored region of the plane,

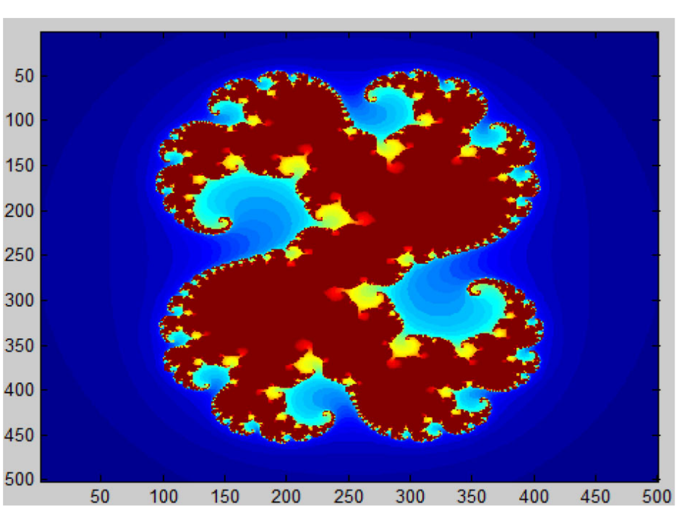

Figure 1. Julia set of $f_\lambda(z)=z^2+\lambda, \lambda=-0.29-0.016$

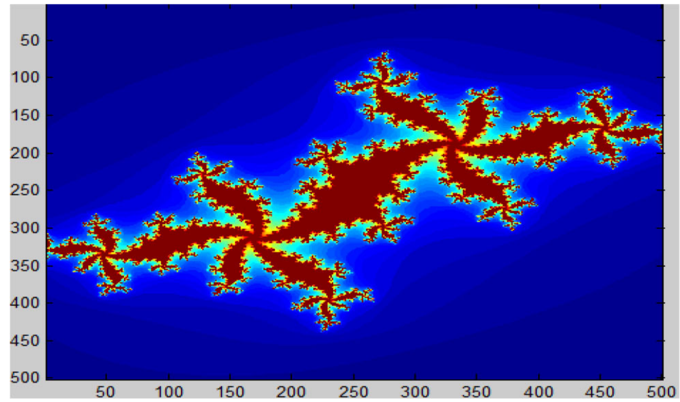

Figure 2. Julia set of $f_\lambda(z)=z^2+\lambda, \lambda=0.5+0.6i$.

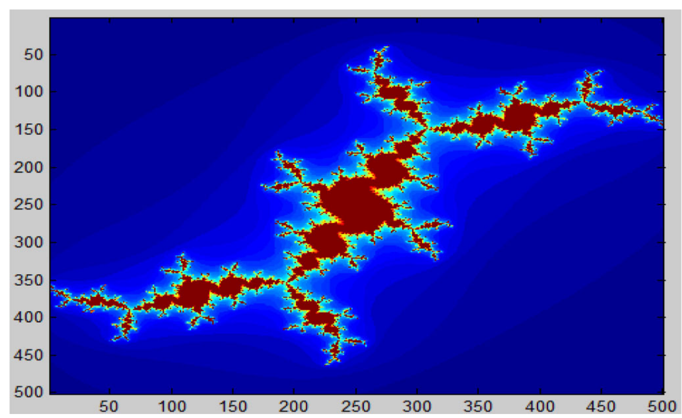

Figure 3. Julia sets of $f_\lambda(z)=z^2+\lambda, \lambda=0.1+0.85i$.

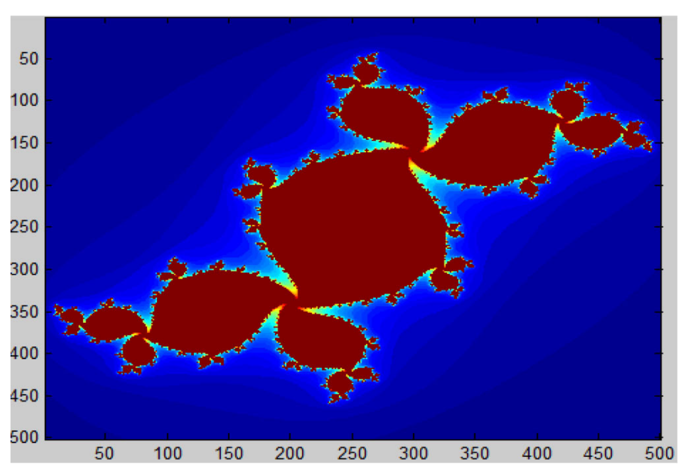

Figure 4. Julia set of $f_\lambda(z)=z^2+\lambda, \lambda=-0.1+0.7i$.

the orbits soon escape to infinity, with red being quite quickly, yellow taking longer, and green taking much longer.

Figures 5 and 6 show how a small change in the value of λ can extremely modify the result. When we look at these shapes they are difficult to describe. One major aspect is that repetitive patterns are seen.

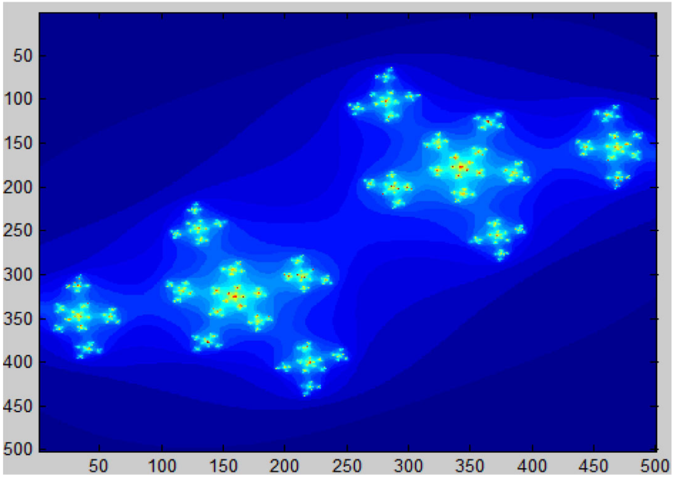

Figure 5. Julia set of $f_\lambda(z) = z^2 + \lambda$, $\lambda = -0.56 + 0.75i$.

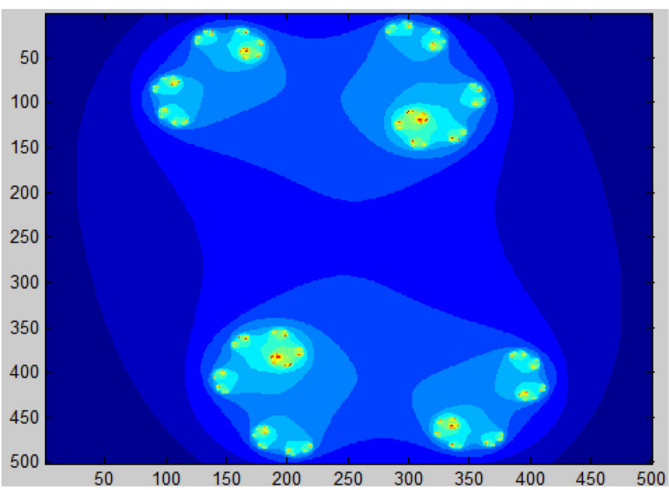

Figure 6. Julia set of $f_\lambda(z) = z^2 + \lambda$, $\lambda = 0.68 - 0.27i$.

The two large "spoke" patterns seen just off-center are repeated off the sides of each spoke, and then are repeated again. If we continue to zoom in on these images, the pattern would appear again and again, a property known as self-similarity.

3.2. Mandelbrot sets for elementary functions

Figure 7 show the Mandelbrot sets of elementary functions $f_\lambda(z) = z^2 + \lambda$.

Figure 7 shows the magnification of the boundary of the cardioid and a primary bulb of the Mandelbrot set.

Figures 7–9 are arguably the most well-known and recognizable fractals ever created. The primary cardioids are located in the center, in a vast black zone. It is bordered on all sides by similar-shaped zones, demonstrating its self-similarity. Figure 8 shows a zoomed-in image along the border of Figure 7 that shows this self-similarity in greater detail.

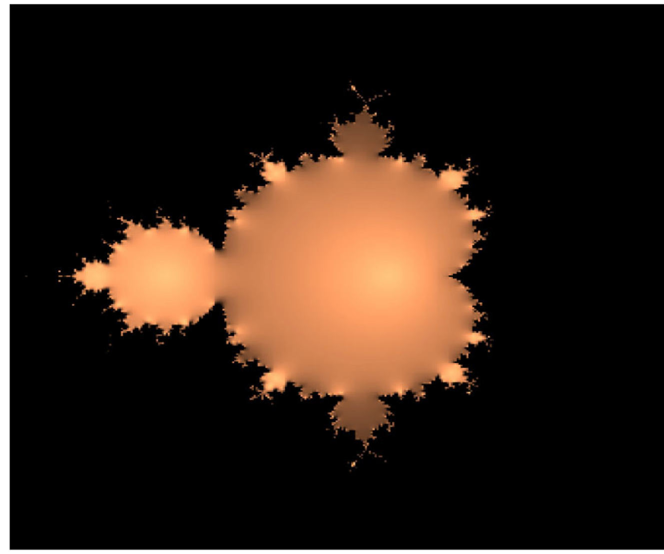

Figure 7. Mandelbrot set for $f_\lambda(z) = z^2 + \lambda$, $\lambda = 0$.

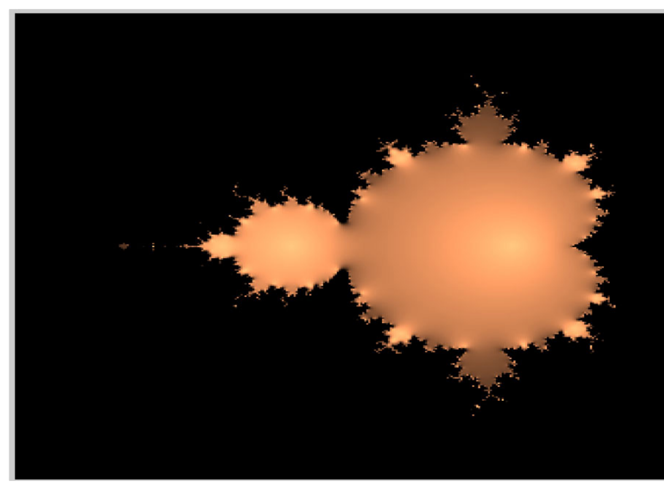

Figure 8. Mandelbrot set for $f_\lambda(z) = z^2 + \lambda$, $\lambda = -0.75$.

3.3. Julia sets for gamma function

For the gamma function, $\Gamma(z)$, we can generate different Julia sets as the value of the parameter λ varies. Figure 10 shows the Julia set for $f_\lambda(z) = \Gamma(z) + \lambda$ within $-6.5 \leq \mathbb{R}(z) \leq 8.5$ and $-8.5 \leq \mathbb{R}(z) \leq 8.5$

Figure 11 shows the Julia set for $f_\lambda(z) = \Gamma(z) + \lambda$ within $-12.0 \leq \mathbb{R}(z) \leq 8.5$ and $-10.5 \leq \mathbb{R}(z) \leq 10.5$

Figure 10 shows the filled-in Julia set for $f_\lambda(z) = \Gamma(z)$. In the figure is a horizontal mast with black spots and spikes according to their proportions, the mast's arms are angled in opposite directions. The basin of attraction of the attractor of the fixed point at $z = 1$ and those of attracting cycles are represented by the black section. The Julia set is the set of complex numbers $z = x + iy$ whose orbits stay constrained after a number of iterations, i.e., the bounds around the mast.

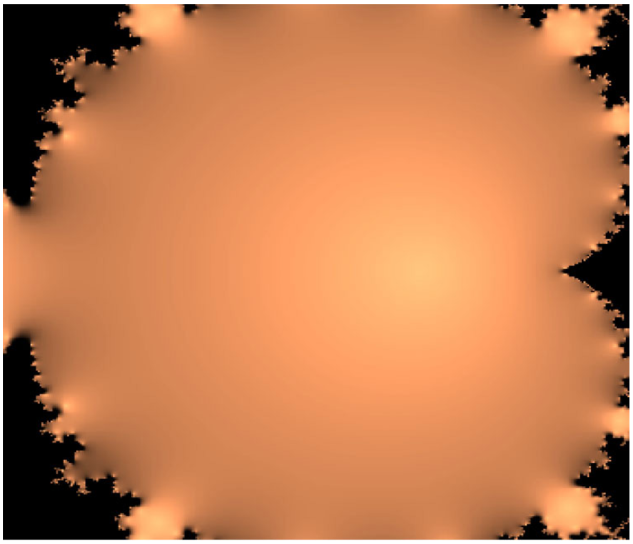

Figure 9. Zoom in on the boundary of Mandelbrot set of $f_\lambda(z) = z^2 + \lambda$.

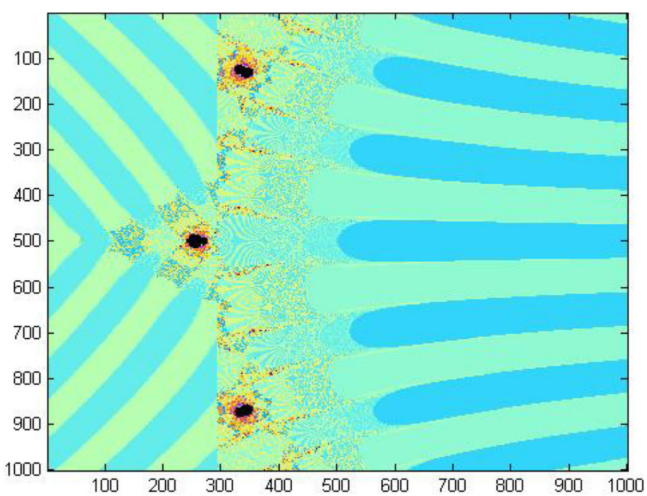

Figure 10. Julia set for $f_\lambda(z) = \Gamma(z) + \lambda$, $\lambda = 0$.

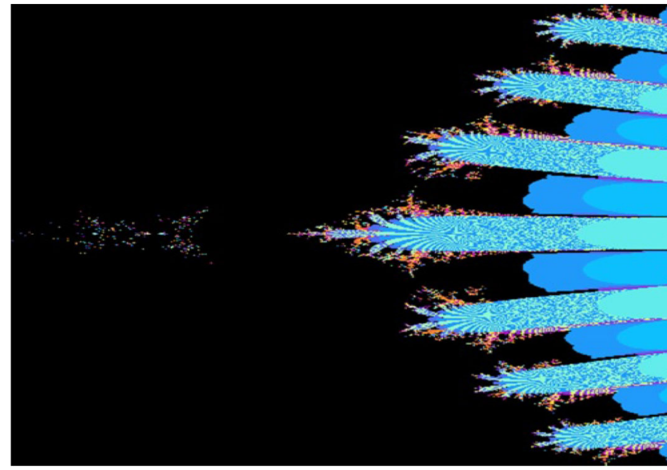

Figure 11. Mandelbrot set for $f_\lambda(z) = \Gamma(z) + \lambda$.

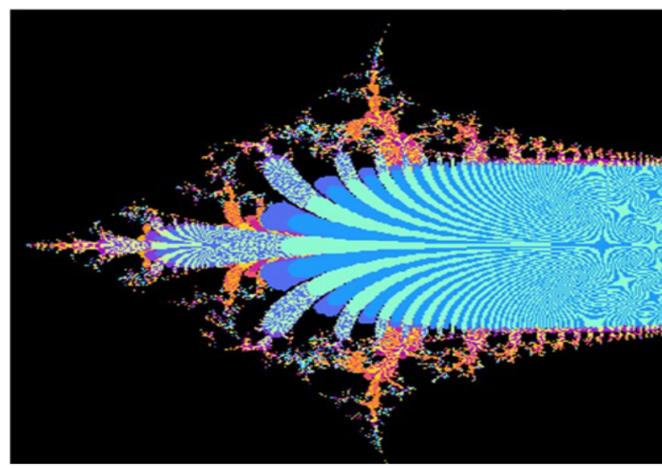

Figure 12. Magnification of the leading frond .

The remaining regions of the plane are colored in various shades to indicate the varying rates at which the point approaches infinity. The masts end around −5 degrees to the left of the real axis, and the remainder of the plane to the left is occupied by the set of $z = x + iy$ whose orbits are in the stable set of infinity.

3.4. Mandelbrot set for gamma function

Figure 11 shows the Mandelbrot set for $f_\lambda(z) = \Gamma(z) + \lambda$ within $-3.5 \leq \mathbb{R}(z) \leq 8.0$ and $-10.0 \leq \mathbb{R}(z) \leq 10.0$.

Figure 12 shows a magnification part of the Mandelbrot set for $f_\lambda(z) = \Gamma(z) + \lambda$ in Figure 11.

For any function that a Julia set generates, there is always a corresponding Mandelbrot set. It is noteworthy that, as infinitely many Julia sets are associated with a function, only one Mandelbrot set is connected to the function. In Figure 11, the black region which forms the bulk of the plane is the basin of attraction of the fixed point and is regarded as the Mandelbrot set for $f_\lambda = \Gamma(z) + \lambda$. In this figure, there are infinitely many "fronds" of different shadings. Each of them is studded with an infinite number of smaller fronds all around its boundary which are miniature copies. Unlike other functions where self-similarity is easily spotted with the Mandelbrot set, the gamma function seems to reveal the fractal of the complement of the Mandelbrot set. This does not imply that the Mandelbrot set has lost its fractal property in the gamma function because the boundary of the complement is also the boundary of the Mandelbrot set. This implies that self-similarity is not only associated with the Mandelbrot set but also its complement.

4. Conclusion

An analysis of the Julia and Mandelbrot sets of some elementary functions was carried out. The results are depicted in the above figures, thereafter the gamma function approximation for Julia and Mandelbrot sets with $\lambda = 0$ are also depicted. The Lanczos beta function approximation is equally presented; we do not illustrate the fractal image of beta function approximation because a high performance computing machine is needed for high iteration. This is an opening for further research.

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