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Compactness with respect to ideal via β-open sets

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ABSTRACT

In this paper, we study β-compactness and present a new class of compactness in ideal topological space, viz. βρI-compact spaces, by utilizing the notion of β-open sets. We also present some basic properties and characterizations of the generalized forms of βρI-compactness and the effect of some different types of functions on them.

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1. Introduction

The concept of ideal in topological space was first studied by Vaidyanathaswamy (1945) and Kuratowski (1966). It was widely studied by Jankovic and Hamlett (1990). Since then, many researchers in the field of General Topology are using ideal to generalize fundamental concepts in General Topology, such as compactness, Lindeloofness, connectedness, Hausdorffness, etc. Newcomb (1967) was the first to introduce the idea of compactness in an ideal topological space and it has been studied and by Racin (1972). Thoroughly, the concept was examined by Hamlett and Jankovic (1990). Thereafter, different classes and strong forms of compactness modulo an ideal have been introduced and studied. Gupta and Kaur (2014) studied and discussed properties of I-compact spaces. They also generalized some of the results in compact spaces, in terms of I-compact spaces. Hamlett et al. (1991) presented and studied countable compactness with respect to an ideal. Abd El-Monsef et al. (1993) used the concept of semi-open sets and introduced SI-compactness in ideal topological spaces. Nasef (2001) introduced another generalized concept

of I-compact spaces via the notion of γ-open sets called γI-compact spaces. Abd El-Monsef et al. (1983) initiated the study of β-open sets in topology, and the analogous notion of semi-preopen set was given exclusively by Andrijevic (1986) and was additionally examined by Ganster and Andrijevic (1988). More applications and backgrounds of β-open sets were discussed by Caldas and Jafari (2009). The aim of this paper is to utilize the notion of β-open sets to examine β-compactness and introduce some types of compactness modulo ideal called βρI-compact. Throughout

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this paper, (X, τ) and (Y, δ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a set $A \subseteq X$, $cl(A)$, $int(A)$, and $X\backslash A$ denote, respectively, the closure, interior, and the complement of A in (X, τ), and $\rho(A)$ denotes the power set of A. The set U_x denotes the set of all open neighborhoods of x in a topological space (X, τ).

2. Preliminaries

Definition: 2.1 An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X such that if:

i) $A\in I$ and $B\subseteq A$, then $B\in I$ (heredity).
ii) $A\in I$ and $B\in I$, then $(A\cup B)\in I$ (finite additivity).

An ideal topological space (or simply ideal space) denoted as (X, τ, I) is a topological space (X, τ) with an ideal I on X. An ideal I in a topological space (X, τ) is said to be:

i) Codense (Dontchev et al., 1999), τ-boundary (Newcomb, 1967), and (Jankovic and Hamlett, 1990), Hayashi-Samuel (Dontchev, 1996) ideal if $I \cap \tau = \{\phi\}$.

ii) σ-ideal if it is countably additive (Jankovic and Hamlett, 1990).

iii) Compatible with τ, denoted by $I \sim \tau$, if for each $A\subseteq X$ and each $a\in A$, there exists $U\in U_a$, such that if $(U\cap A)\in I$, then $A\in I$ (Jankovic and Hamlett, 1992).

Definition: 2.2 Given an ideal space (X, τ, I) and a subset A of X, a set operator $(.)^*$: $P(X) \to P(X)$ called a local function of A with respect to τ and I is defined to be the set $A^*(I, \tau) = \{x \in X: (A\cap U) \notin I$, for every $U \in U_x\}$. The Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(I, \tau)$, called the *–topology, finer than τ, is defined by $Cl^*(A) = A\cup A^*(I, \tau)$. Where no confusion will arise, we write A^* for $A^*(I, \tau)$ and τ^* for $\tau^*(I, \tau)$. For an ideal topological space (X, τ, I), the collection $\{U\backslash A: U \in \tau$ and $A \in I\}$ is a base for τ^*. A subset A of X is said to be τ^*-closed if $A^*\subseteq A$ (Jankovic and Hamlett, 1990). A subset A of X is said to be τ^*-perfect if $A^* = A$ and it is said to be τ^*-dense if $A \subseteq A^*$ (Hayashi, 1964).

Example: 2.3 For a given topological space (X, τ), the following are some of the important ideals on (X, τ): $I = \{\phi\}$, $I_f = \{A \subseteq X : A \text{ is finite}\}$, $I_c = \{A \subseteq X : A \text{ is countable}\}$, $\rho(X)$, and $P(U) = \{A \subseteq U : U \subseteq X\}$ for any given subset U of X.

Lemma: 2.4 If I and J are ideals on a topological space (X, τ), then the following hold:

a) $I\cap J=\{A:A\in I$ and $A\in J\}$ is an ideal on X.

b) $I\vee J=\{A\cup B$: $A\in I$ and $B\in J\}$ is an ideal on X (Jankovic and Hamlett, 1990).

Lemma: 2.5 If I and J are ideals on a topological space (X, τ), then $I\wedge J = \{A\cap B$: $A\in I$ and $B\in J\}$ is an ideal on X.

Proof

i) $E\in I\wedge J$ and $F\subseteq E \Rightarrow F\subseteq E=A\cap B$ for some $A\in I$ and $B\in J$ $\Rightarrow \exists\, A_1\subseteq A$ and $\exists B_1\subseteq B$ such that $F=A_1\cap B_1 \subseteq E=A\cap B$. Now $A_1\subseteq A$ and $B_1\subseteq B \Rightarrow A_1\in I$ and $B_1\in J$. Hence, $F=A_1 \cap B_1$ for some $A_1\in I$ and $B_1\in J$, i.e., $F\in I\wedge J$.

ii) $E\in I\wedge J$ and $F\in I\wedge J \Rightarrow E=A_1\cap B_1$ and $F=A_2\cap B_2$ $\Rightarrow E\subseteq A_1$, $E\subseteq B_1$ and $F\subseteq A_2$, $F\subseteq B_2$ for some $A_1,A_2\in I$ and $B_1,B_2\in J \Rightarrow E\cup F\subseteq A_1\cup A_2=A_3$ and $E\cup F\subseteq B_1\cup B_2=B_3$ for some $A_3\in I$ and $B_3\in J$. Hence, $E\cup F\in I\wedge J$. i.e. $E\cup F=G$ for some $G\in I$ and $G\in J$.

Lemma: 2.6 $I\cap J = I\wedge J$.

Proof $A\in I\cap J \Rightarrow A\in I$ and $A\in J \Rightarrow A\cap A=A\in I\wedge J$.

Conversely, $A\in I\wedge J \Rightarrow A=E\cap F$ for some $E\in I$ and $F\in J$

$\Rightarrow A\subseteq E \in I$ and $A\subseteq F\in J \Rightarrow A\in I$ and $A\in J \Rightarrow A\in I\cap J$.

Lemma: 2.7 a) $I \wedge J \subseteq I$ and $I\wedge J \subseteq J$ b) $I\subseteq I\vee J$ and $J\subseteq I\vee J$.

Proof

a) $A\in I\wedge J \Rightarrow A=E\cap F$ for some $E\in I$ and $F\in J \Rightarrow A\subseteq E\in I \Rightarrow A\in I$.

b) Let $A\in I$, then $\forall B\in J$, $A\subseteq A\cup B\in I\vee J$. Now, $I\vee J$ is an ideal, $A\cup B\in I\vee J$, $\forall B\in J$, and $A\subseteq (A\cup B)$, and $\forall B\in J \Rightarrow A\in I\vee J$.

Definition: 2.8 A subset A of a space (X, τ) is said to be β-open (Abd El-Monsef, 1983) [pre-semi-open (Andrijevic, 1986)] if $A \subseteq cl(int(cl(A)))$. The complement of a β-open set is called a β-closed set. The collection of all β-open sets in a topological space (X, τ) is denoted as $\beta O(X, \tau)$.

Definition: 2.9 The β-closure of A is the intersection over all β-closed sets containing A and is denoted by $\beta cl(A)$. The β-interior of A is the union over all β-open sets contained in A and is denoted by $\beta int(A)$.

Lemma: 2.10 For a topological space (X, τ), the following hold:

a) $\tau\subseteq\beta O(X,\tau)$.

b) $\beta O(X,\tau)$ is closed under arbitrary union.

c) $\beta O(X,\tau) \subseteq \beta O(X,\tau^*)$.

Proof

a) (X,τ) is a topological space and $A\in\tau \Rightarrow A\subseteq cl(A)$ $\Rightarrow A=int(cl(A)) \Rightarrow A\subseteq cl(int(cl(A)))$ $\Rightarrow A \in\beta O(X,\tau)$.

b) Consider the collection $\{A_\alpha:\alpha\in\Delta\}$ and let $A_\alpha\in\beta O(X,\tau),\forall\alpha\in\Delta$. Then, $A_\alpha\subseteq cl(\text{int}(cl(A)))$, $\forall\alpha\in\Delta$. and $\bigcup_{\alpha\in\Delta}A_\alpha\subseteq\bigcup_{\alpha\in\Delta}[cl(\text{int}(cl(A_\alpha)))]$ $\subseteq cl[\bigcup_{\alpha\in\Delta}(\text{int}(cl(A_\alpha)))]\subseteq cl(\text{int}(\bigcup_{\alpha\in\Delta}(cl(A_\alpha))))\subseteq$ $cl(\text{int}(cl(\bigcup_{\alpha\in\Delta}A_\alpha)))$.
Setting $Q=\bigcup_{\alpha\in\Delta}A_\alpha$ we have $Q\subseteq cl(\text{int}(cl(Q)))$.

c) τ^* is finer than τ, i.e., $\tau\subseteq\tau^*$ implies $\beta O(X,\tau)\subseteq\beta O(X,\tau^*)$.

Remark: 2.11 By (a) of Lemma 2.10 $\tau\subseteq\beta O(X, \tau) \Rightarrow$ every open set is β-open ⇒ compliment of every open set is β-closed ⇒ every closed set is β-closed.

Definition: 2.12 A subset A of an ideal space (X, τ, I) is said to β-open with respect to I (or βI-open) if there exists an open set U such that

i) $(U\backslash A)\in I$, and

ii) $(A\backslash cl(int(cl(U))))\in I$ (Catalan et al., 2019).

Definition: 2.13 A subset A of an ideal space (X, τ, I) is said to be generalized closed with respect to ideal I (briefly Ig-closed) if and only if $(cl(A)\backslash U)\in I$, whenever $A\subseteq U$ and $U\in\tau$ (Catalan et al., 2019).

Definition: 2.14 A subset A of an ideal space (X, τ, I) is said to be Igβ-closed if and only if $(\beta cl(A)\backslash U)\in I$, whenever $A\subseteq U$ and $U\in\beta O(X,\tau)$.

Definition: 2.15 A subset A of a space (X, τ) is said to be generalized β-closed (gβ-closed) if $\beta cl(A)\subseteq U$, whenever $A\subseteq U$ and $U\in\beta O(X,\tau)$.

Lemma: 2.16 Every gβ-closed set in an ideal space $(X, \tau, \{\phi\})$ is $\{\phi\}$gβ-closed.

Proof Let $(X,\tau,\{\phi\})$ be an ideal space and A be gβ-closed and $A\subseteq U$, $U\in\beta O(X,\tau)$. Now, $A\subseteq U$ implies $A\backslash U = \phi\in\{\phi\}$ and A is gβ-closed $\Rightarrow$ $(\beta cl(A)\backslash U)\in I=\{\phi\}\Rightarrow$ $\beta cl(A)\subseteq U$. Hence, A is $\{\phi\}$gβ-closed.

Corollary: 2.17 Every gβ-closed set in an ideal space $(X, \tau, \rho(X))$ is $\rho(X)$gβ-closed.

Definition: 2.18 A function $f:(X,\tau)\rightarrow(Y,\delta)$ is said to be:

a) β-irresolute, if $\forall U\in\beta O(Y,\delta), f^{-1}(U)\in\beta O(X,\tau)$,

b) Mβ-open, if $\forall U\in\beta O(X,\tau), f(U)\in\beta O(Y,\delta)$ (Mahmoud and Abd El-Monsef, 1990).

Lemma: 2.19 If $f:(X,\tau)\rightarrow(Y,\delta)$ is β-irresolute surjection and I is an ideal on X, then

a) $J=\{A\subseteq Y: f^{-1}(A)\in I\}$ is ideal on Y.

b) $f(I)=\{f(A):A\in I\}$ is ideal on Y (Newcomb, 1967).

Lemma: 2.20 If $f:(X,\tau)\rightarrow(Y,\delta)$ is Mβ-open injection and J is an ideal on Y, then $f^{-1}(J)=\{f^{-1}(A):A\in J\}$ is ideal on X (Mahmoud and Abd El-Monsef, 1990).

Definition: 2.21 A topological space (X, τ) is said to be:

a) β-Hausdorff or (βT_2)-space if given any two distinct points $x,y\in X$ there exist disjoint β-open sets U and V such that $x\in U$ and $y\in V$ (Mahmoud and Abd El-Monsef, 1990).

b) B-Urysohn or $(\beta T_{2\frac{1}{2}})$-space if given any two distinct points $x,y\in X$ there exist disjoint β-open sets U and V such that $x\in U$ and $y\in V$ and $\beta cl(U)\cap\beta cl(V)=\phi$.

c) β-normal if given any two disjoint closed sets A and B in X there exist disjoint β-open sets U and V such that $A\subseteq U$ and $B\subseteq V$ (Mahmoud and Abd El-Monsef, 1990).

Lemma: 2.22 Every β-Urysohn space is β-Hausdorff

Proof (X, τ) is $(\beta T_{2\frac{1}{2}})$-space ⇒ given any two distinct points $x,y\in X$, there exist disjoint β-open sets U and V such that $x\in U$ and $y\in V$, and $\beta cl(U)\cap\beta cl(V)=\phi$ ⇒ given any two distinct points $x,y\in X$, there exist two β-open sets U and V such that $x\in U$ and $y\in V$ and $U\cap B=\phi$ ⇒ (X, τ) is (βT_2)-space.

Lemma: 2.23 Every completely β-normal space is β-normal

Definition: 2.24 Let (X, τ) be a topological space. A collection $\{U_\alpha:\alpha\in\Delta\}$ of subsets of X is said to be:

a) A cover for X if $X\subseteq\bigcup_{\alpha\in\Delta}U_\alpha$.

b) An open (resp., β-open) cover for X if $\{U_\alpha:\alpha\in\Delta\}$ is a cover for X and $\forall\alpha\in\Delta, U_\alpha\in\tau$ (resp., $U_\alpha\in\beta O(X,\tau)$) (Mahmoud and Abd El-Monsef, 1990).

Definition: 2.25 A topological space (X, τ) is said to be

i) β-compact (resp. countably β-compact) if for every β-open (resp. countable β-open) cover $\{U_\alpha:\alpha\in\Delta\}$ for X, $\exists\Delta_o\subseteq\Delta$ finite such that $X\subseteq\bigcup_{\alpha\in\Delta_o}U_\alpha$ (Mahmoud and Abd El-Monsef, 1990).

ii) β-compactum if (X,τ) is β-compact and β-Hausdorff.

Definition: 2.26 An ideal space (X, τ, I) is said to be

i) I-compact (resp. countably I-compact) if for every open (resp. countable open) cover $\{U_\alpha : \alpha \in \Delta\}$ for X, $\exists \Delta_o \subseteq \Delta$ finite such that $(X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in I$. (Hosny, 2013).

ii) βI-compact (resp. countably βI-compact) if for every β-open (resp. countable β-open) cover $\{U_\alpha : \alpha \in \Delta\}$ for X, $\exists \Delta_o \subseteq \Delta$ finite such that $(X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in I$. (Mahmoud and Abd El-Monsef, 1990).

3. β-Compactness

Theorem 3.1 If (X, τ) is β-compact and B is β-closed subset of X, then B is β-compact.

Proof Let (X, τ) be β-compact, B be β-closed subset of X, and $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets such that $B \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$. Then, ($X$\B) is β-open and $X \subseteq \bigcup_{\alpha\in\Delta} U_\alpha \cup (X \setminus B)$. Since X is β-compact, $\exists \Delta_o \subseteq \Delta$ finite such that $B \subseteq X \subseteq \bigcup_{\alpha\in\Delta_o} U_\alpha \cup (X \setminus B)$. Hence, $B \subseteq \bigcup_{\alpha\in\Delta_o} U_\alpha$. Therefore, B is β-compact.

Lemma: 3.2 If (X, τ) is β-compact and B is closed subset of X, then B is β-compact.

Theorem 3.3 If B is a β-compact set in a β-Hausdorff space X and $x \notin B$, then there exist β-open sets U and V such that $x \in U, B \subseteq V$ and $U \cap V = \phi$.

Proof Let B be a β-compact set in a β-Hausdorff space X and $x \notin B$. Now, $\forall b \in B, x \neq b$ and since X is β-Hausdorff, there exist β-open sets U_x and U_b such that $x \in U_x$, $b \in U_b$, and $U_x \cap U_b = \phi$. The family $\{V_b : b \in V\}$ forms a β-open cover for B. Since B is β-compact, there exist a finite subcollection $\{V_{b_i} : i = 1,2,\ldots,n\}$ such that $B \subseteq \bigcup_{i=1}^{n} V_i$. Let $V = \bigcup_{i=1}^{n} V_i$, then V is β-open and B⊆V. For each $V_{b_i}, i = 1,2,\ldots,n$ there exists a corresponding U_i such that $x \in U_i$ and $U_i \cap V_i = \phi$. Let $U = \beta\operatorname{int}(\bigcap_{i=1}^{n} U_i)$, then U is β-open, $x \in U, B \subseteq V$, and $U \cap V = \phi$.

Theorem 3.4 A β-compact set in a β-Hausdorff space is β-closed.

Proof Let X be β-Hausdorff and $B \subseteq X$ be β-compact. Let $x \notin B$, then by Theorem 3.3, there exist disjoint β-open sets U and V such that $x \in U, B \subseteq V$. This implies $\forall x \notin B$, there exist β-open set U_x such that $x \in U_x$ and $U_x \cap B = \phi$. Let $H = \bigcup\{U_x : x \notin B\}$, then H is b-open and ($X/H$) = B. Hence, B is β-closed.

Theorem 3.5 Every β-compactum is β-normal.

Proof Let X be β-compact and β-Hausdorff and A and B be any disjoint closed sets in X. Then, A and B are β-closed and disjoint. By Theorem 3.1, A and B are β-compact and $\forall a \in A, a \notin B.$. Therefore, by Theorem 3.3, there exist β-open sets U_a and V_a with $a \in U_a$ and $B \subseteq V_a$ such that $U_a \cap V_a = \phi$. Now, $A \subseteq \bigcup_{a\in A} U_a$ and $B \subseteq V_a, a \in A$. Since A is β-compact, there exists a finite subcover say $\{U_{a_i} : i = 1,2,\ldots,n\}$ such that $A \subseteq \bigcup_{i=1}^{n} U_{a_i}$. Let $U = \bigcup_{i=1}^{n} U_{a_i}$, then U is β-open and A⊆U. Now, for each U_{a_i} there exists V_{a_i} such that $B \subseteq V_{a_i}$ and $U_{a_i} \cap V_{a_i} = \phi$. Let $V = \beta\operatorname{int}(\bigcap_{i=1}^{n} V_{a_i})$, then V is β-open and B⊆V. Moreover, $U \cap V = \phi$. Hence, X is β-normal.

Definition: 3.6 An ideal space (X, τ, I) is said to be Igβ-normal; if given any two disjoint Igβ-closed sets A and B there exist two disjoint β-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 3.7 If (X, τ, I) is an ideal space and (X, τ) is β-completely normal, then (X, τ, I) is Igβ-normal.

Proof Let (X, τ) be β-completely normal, A and B be any two disjoint Igβ-closed sets. (X, τ) is completely β-normal and implies that there exist disjoint β-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. This implies that (X, τ, I) is Igβ-normal.

Theorem 3.8 If X is β-compactum, then $A \subseteq X$ is β-closed if and only if A is β -compact.

Proof X is β-compactum and $A \subseteq X$ is β-closed ⇒ A is β-compact (Theorem 3.1).

Conversely, X is a β-compactum and $A \subseteq X$ is β-compact ⇒A is β-closed (Theorem 3.4).

Theorem 3.9 A space (X, τ) is β-compact if and only if every family $\{U_\alpha : \alpha \in \Delta\}$ of β-closed sets in X having the finite intersection property has a nonempty intersection.

Proof Let X be β-compact and $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-closed sets in X having the finite intersection property.

Suppose $\bigcap_{\alpha\in\Delta} U_\alpha = \phi$, then $\bigcup_{\alpha\in\Delta} (X \setminus U_\alpha) = X \setminus \bigcap_{\alpha} U_\alpha = X \setminus \phi = X$. This implies $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a β-open cover for X. Since X is β-compact, there is a finite subcover say $\{X \setminus U_{\alpha_i} : i = 1,2,\ldots,n\}$ such that $X \subseteq \bigcup_{i=1}^{n} (X \setminus U_{\alpha_i}) = X \setminus \bigcap_{i=1}^{n} U_{\alpha_i}$. This implies $\bigcap_{i=1}^{n} U_{\alpha_i} = \phi$, and this a contradiction, i.e., $\{U_\alpha : \alpha \in \Delta\}$ has the finite intersection property. Hence, $\bigcap_{\alpha\in\Delta} U_\alpha \neq \phi$.

Conversely, suppose the condition holds and let $\{U_\alpha : \alpha \in \Delta\}$ be a β-open cover for *X*. Then, $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of β-closed subsets of *X* such that $\bigcap_{\alpha\in\Delta}(X \setminus U_\alpha) = X \setminus \bigcup_{\alpha\in\Delta} U_\alpha = X \setminus X = \phi$. By our hypothesis, the collection $\{X \setminus U_\alpha : \alpha \in \Delta\}$ has no finite intersection property. Therefore, there is a finite subcollection such that $\bigcap_{i=1}^{n}(X \setminus U_{\alpha_i}) = X \setminus \bigcup_{i=1}^{n} U_{\alpha_i} = \phi$. This implies $X \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. Hence, the collection $\{U_{\alpha_i} : i = 1, 2, \ldots, n\}$ is a finite sub-cover of $\{U_\alpha : \alpha \in \Delta\}$ for *X*. Therefore, *X* is β-compact.

4. βρI-Compactness

Definition: 4.1 A subset A of an ideal space (*X*, *τ*, I) is said to be ρI-compact if for every family $\{U_\alpha : \alpha \in \Delta\}$ of open subsets of *X*, if $(A \setminus \bigcup_{\alpha\in\Delta} U_\alpha) \in I$, $\exists \Delta_o \subseteq \Delta$ finite such that $(A \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in I$. The space (*X*, *τ*, I) is said to be ρI-compact if *X* is ρI-compact (Pachon, 2016).

Definition: 4.2 A subset A of an ideal space (*X*, *τ*, I) is said to be βρI-compact if for every family $\{U_\alpha : \alpha \in \Delta\}$ of β-open sets in *X*, if $(A \setminus \bigcup_{\alpha\in\Delta} U_\alpha) \in I$, $\exists \Delta_o \subseteq \Delta$ finite such that $(A \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in I$. The space (*X*, *τ*, I) is said to be βρI-compact if *X* is βρI-compact.

Example: 4.3

i) Let $X = \mathbb{R}$ be equipped with the standard topology, and I =ρ(*R*) be an ideal on $\mathbb{R}$. Then, the ideal space $(\mathbb{R}, \tau, \rho(\mathbb{R}))$ is βρI-compact, i.e., if $\{U_\alpha : \alpha \in \Delta\}$ a family of β-open sets in $\mathbb{R}$, if $(\mathbb{R} \setminus \bigcup_{\alpha\in\Delta} U_\alpha) \in \rho(\mathbb{R})$, $\exists \Delta_o \subseteq \Delta$ finite such that $(\mathbb{R} \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in \rho(\mathbb{R})$.

ii) Let $X = \{x, y, z\}$, $\tau = \{\phi, X, \{x\}, \{x, z\}\}$ and $I = I_f$ Then, the space (*X*, *τ*, I) is βρI-compact, i.e., for any collection $\{U_\alpha : \alpha \in \Delta\}$ of members of $\beta o(x, \tau) = \{\phi, X, \{x\}, \{x, y\}, \{x, z\}\}$, for which $(X \setminus \bigcup_{\alpha\in\Delta} U_\alpha) \in I_f$, $\exists \Delta_o \subseteq \Delta$ finite such that $(X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in I_f$.

Lemma: 4.4 If $(X, \tau, \{\phi\})$ is $\beta\rho\{\phi\}$-compact and B is β-closed in *X*, then B is $\beta\rho\{\phi\}$-compact.

(Theorem 3.1).

Lemma: 4.5 If *X* is $\beta\rho\{\phi\}$-compact and β-Hausdorff, then *X* is β-normal (Theorem 3.5).

Theorem 4.6 A topological space (*X*, *τ*) is β-compact if and only if $(X, \tau, \{\phi\})$ is $\beta\rho\{\phi\}$-compact.

Proof Let (*X*, *τ*) be β-compact. This implies given any collection $\{U_\alpha : \alpha \in \Delta\}$ of β-open sets in *X* such that $(X \setminus \bigcup_{\alpha\in\Delta} U_\alpha) = \phi$, $\exists \Delta_o \subseteq \Delta$ finite such that $(X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) = \phi \in \{\phi\}$. This implies $(X, \tau, \{\phi\})$ is $\beta\rho\{\phi\}$-compact.

Conversely, let $(X, \tau, \{\phi\})$ be $\beta\rho\{\phi\}$-compact and $\{U_\alpha : \alpha \in \Delta\}$ be any β-open cover for *X*. This implies $(X \subseteq \bigcup_{\alpha\in\Delta} U_\alpha) \Rightarrow X \setminus \bigcup_{\alpha\in\Delta} U_\alpha = \phi \in \{\phi\} \Rightarrow \exists \Delta_o \subseteq \Delta$ finite such that $(X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in \{\phi\} \Rightarrow (X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) = \phi \Rightarrow X \subseteq \bigcup_{\alpha\in\Delta_o} U_\alpha \Rightarrow$ (*X*, *τ*) is β-compact.

Lemma: 4.7 If (*X*, *τ*, I) is βρI-compact for every ideal I, then (*X*, *τ*) is β-compact.

Theorem 4.8 If (*X*, *τ*, I) is βρI-compact, then (*X*, *τ*, I) is ρI-compact.

Proof Let (*X*, *τ*, I) be βρI-compact and $\{U_\alpha : \alpha \in \Delta\}$ be a family of open sets in *X* such that $(X \setminus \bigcup_{\alpha\in\Delta} U_\alpha) \in I$. By (*a*) of Lemma 2.10 $U_\alpha \in \beta o(X, \tau), \forall \alpha \in \Delta$. By our hypothesis, $\exists \Delta_o \subseteq \Delta$ finite such that $(X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in I$. Hence, (*X*, *τ*, I) is ρI-compact.

Theorem 4.9 An ideal space (*X*, *τ*, I) is βρI-compact iff (*X*, *τ**, I) is βρI-compact.

Proof Suppose (*X*, *τ*, I) is βρI-compact and $\{U_\alpha : \alpha \in \Delta\}$ is a collection of β-open sets in (*X*, *τ**,I) such that $(X \setminus \bigcup_{\alpha\in\Delta} U_\alpha) \in I$. Now, $U_\alpha = V_\alpha \setminus W_\alpha$ such that $V_\alpha \in \beta o(X, \tau), W_\alpha \in I, \forall \alpha \in \Delta$. Therefore, $\{V_\alpha : \alpha \in \Delta\}$ is a collection of β-open sets in (*X*, *τ* ,I) and $\bigcup_{\alpha\in\Delta} U_\alpha \subseteq \bigcup_{\alpha\in\Delta} V_\alpha$. Therefore, $X \setminus \bigcup_{\alpha\in\Delta} V_\alpha \subseteq X \setminus \bigcup_{\alpha\in\Delta} U_\alpha \in I$ implies $X \setminus \bigcup_{\alpha\in\Delta} V_\alpha \in I$. By our hypothesis $\exists \Delta_o \subseteq \Delta$ finite such that $(X \setminus \bigcup_{\alpha\in\Delta_o} V_\alpha) \in I$. Now, $X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \subseteq [(X \setminus \bigcup_{\alpha\in\Delta_o} V_\alpha) \cup (\bigcup_{\alpha\in\Delta_o} W_\alpha)] \in I$ $\Rightarrow (X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in I$. This implies (*X*, *τ**,I) is βρI-compact.

Conversely, suppose (*X*, *τ**,I) is βρI-compact and $\{U_\alpha : \alpha \in \Delta\}$ is a collection of β-open sets in

(*X*, *τ*, I) such that $(X \setminus \bigcup_{\alpha\in\Delta} U_\alpha) \in I$. By (*c*) of Lemma 2.10 and our hypothesis, $\{U_\alpha : \alpha \in \Delta\}$ is a collection of β-open sets in (*X*, *τ**,I) and $\exists \Delta_o \subseteq \Delta$ finite such that $(X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in I$. Hence, (*X*, *τ*, I) is βρI-compact.

Theorem 4.10 An ideal space (*X*, *τ*, I) is βρI-compact iff for any family $\{V_\alpha : \alpha \in \Delta\}$ of β-closed sets in *X*, if $\bigcap_{\alpha} V_\alpha \in I$, then $\exists \Delta_o \subseteq \Delta$ finite such that $\bigcap_{\alpha\in\Delta_o} V_\alpha \in I$.

Proof Let (X, τ, I) be a βρI-compact and $\{V_\alpha : \alpha \in \Delta\}$ be a family of β-closed sets in X such that $\bigcap_\alpha V_\alpha \in I$. Now, $U_\alpha = X \setminus V_\alpha, \forall \alpha \in \Delta$, then $\{U_\alpha : \alpha \in \Delta\}$ is a family of β-open sets in X. Therefore, we have $\bigcap_\alpha V_\alpha = \bigcap_{\alpha \in \Delta}(X \setminus U_\alpha) = X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. By our hypothesis, $\exists \Delta_o \subseteq \Delta$ finite such that $(X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) \in I$. Hence, $(X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) = \bigcap_{\alpha \in \Delta_o}(X \setminus U_\alpha) = \bigcap_{\alpha \in \Delta_o} V_\alpha \in I$.

Conversely, suppose the condition holds and $\{U_\alpha : \alpha \in \Delta\}$ is a family of β-open sets in X such that $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. Then $\{(X \setminus U_\alpha) : \alpha \in \Delta\}$ is a family of β-closed sets in X and $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha = \bigcap_{\alpha \in \Delta}(X \setminus U_\alpha) \in I$. By our hypothesis, $\exists \Delta_o \subseteq \Delta$ finite such that $\bigcap_{\alpha \in \Delta_o}(X \setminus U_\alpha) = (X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) \in I$. Let (X, τ, I) be a βρI-compact.

Corollary: 4.11 (X, τ^*, I) is βρI-compact if and only if for any family $\{V_\alpha : \alpha \in \Delta\}$ of β-closed sets in (X, τ^*, I), if $\bigcap_\alpha V_\alpha \in I$, then $\exists \Delta_o \subseteq \Delta$ finite such that $\bigcap_{\alpha \in \Delta_o} V_\alpha \in I$.

Theorem 4.12 Let I and J be ideals in $(X; \tau)$ and $K = I \cap J$. If (X, τ, I) is βρI-compact and

(X, τ, J) is βρJ-compact, then (X, τ, K) is βρK-compact.

Proof Suppose the condition holds and let $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets in X such that $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in K$. This implies $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$ and $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in J$. Now, if (X, τ, I) is βρI-compact and (X, τ, J) is βρJ-compact, then there exist finite subsets Δ_1 and Δ_2 of Δ such that $X \setminus \bigcup_{\alpha \in \Delta_1} U_\alpha \in I$. and $X \setminus \bigcup_{\alpha \in \Delta_2} U_\alpha \in J$. Taking $\Delta_3 = \Delta_1 \cap \Delta_2$, then Δ_3 is finite and $X \setminus \bigcup_{\alpha \in \Delta_3} U_\alpha \in I \cap J = K$. Hence, (X, τ, K) is βρK-compact.

Theorem 4.13 If A and B are βρI-compact subsets of an ideal space (X, τ, I), then $(A \cup B)$ is βρI-compact.

Proof Let A and B be βρI-compact subsets of an ideal topological space (X, τ, I) and $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets in X such that $(A \cup B) \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. This implies $A \setminus \bigcup_{\alpha \in \Delta} U_\alpha \subseteq (A \cup B) \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$ and similarly $B \setminus \bigcup_{\alpha \in \Delta} U_\alpha \subseteq (A \cup B) \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. Since A and B are βρI-compact, there exist finite subsets Δ_1 and Δ_2 of Δ such that $A \setminus \bigcup_{\alpha \in \Delta_1} U_\alpha \in I$ and $B \setminus \bigcup_{\alpha \in \Delta_2} U_\alpha \in I$. Taking $\Delta_3 = \Delta_1 \cup \Delta_2$, then Δ_3 is finite and we have $A \setminus \bigcup_{\alpha \in \Delta_3} U_\alpha \in I$ and $B \setminus \bigcup_{\alpha \in \Delta_3} U_\alpha \in I$. This implies $(A \cup B) \setminus \bigcup_{\alpha \in \Delta_3} U_\alpha \in I$. Hence, $(A \cup B)$ is βρI-compact.

Lemma: 4.14 Finite union of βρI-compact sets is βρI-compact.

Theorem 4.15 Let A be a βρI-compact subset of an ideal space (X, τ, I). If B is a β-open set and $B \subseteq A$, then A\B is βρI-compact.

Proof Let A be a βρI-compact subset of an ideal topological space (X, τ, I) and B be a β-open set such that $B \subseteq A$. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets in X such that $(A \setminus B) \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. This implies $A \setminus \bigcup_{\alpha \in \Delta}(B \cup U_\alpha) \in I$. Given that A is a βρI-compact and B is β-open, $\exists \Delta_o \subseteq \Delta$ finite such that $A \setminus \bigcup_{\alpha \in \Delta_o}(B \cup U_\alpha) \in I$. This implies $(A \setminus B) \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \in I$. Hence, $(A \setminus B)$ is βρI-compact.

Theorem 4.16 Let (X, τ, I) be an ideal space and $A \subseteq X$. If for every collection $\{U_\alpha : \alpha \in \Delta\}$ of β-open sets, if $(A \setminus \bigcup_{\alpha \in \Delta} U_\alpha) \in I$ there exist a βρI-compact set B containing A such that $(B \setminus \bigcup_{\alpha \in \Delta} U_\alpha) \in I$, then A is βρI-compact.

Proof Suppose the condition holds. Let $A \subseteq X$, $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets such that $(A \setminus \bigcup_{\alpha \in \Delta} U_\alpha) \in I$ and B be βρI-compact set containing A such that $(B \setminus \bigcup_{\alpha \in \Delta} U_\alpha) \in I$. Now, B is βρI-compact implies $\exists \Delta_o \subseteq \Delta$ finite such that $(B \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) \in I$. And, $A \subseteq B \Rightarrow (A \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) \in I$. Hence, A is βρI-compact.

Theorem 4.17 If $(X, \tau, \{\phi\})$ is βρ{ϕ}-compact, then every ϕgβ-closed subset of $(X, \tau, \{\phi\})$ is β-compact.

Proof Let A be a ϕgβ-closed subset of βρ{ϕ}-compact space $(X, \tau, \{\phi\})$ and $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets such that $(A \subseteq \bigcup_{\alpha \in \Delta} U_\alpha)$. Now, A is ϕgβ-closed $\Rightarrow \beta cl(A) \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. Therefore, $X = (X \setminus \beta cl(A)) \cup \bigcup_{\alpha \in \Delta} U_\alpha \Rightarrow X \setminus [(X \setminus \beta cl(A)) \cup \bigcup_{\alpha \in \Delta} U_\alpha] = \phi \in \{\phi\}$. Since $X \setminus \beta cl(A)$ is β-open and $(X, \tau, \{\phi\})$ is βρ{ϕ}-compact, then $\exists \Delta_o \subseteq \Delta$ finite such that $X \setminus [(X \setminus \beta cl(A)) \cup \bigcup_{\alpha \in \Delta_o} U_\alpha] = \phi$. But $X \setminus [(X \setminus \beta cl(A)) \cup \bigcup_{\alpha \in \Delta_o} U_\alpha] = \beta cl(A) \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha$. This implies $A \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha = \phi \Rightarrow (A \subseteq \bigcup_{\alpha \in \Delta_o} U_\alpha)$. Hence, A is β-compact.

Theorem 4.18 If A and B are subsets of an ideal space $(X, \tau, \{\phi\})$ such that $A \subseteq B, B \subseteq \beta cl(A)$ and A is {ϕ}gβ-closed , then A is βρ{ϕ}-compact if and only if B is βρ{ϕ}-compact

Proof Suppose the condition holds and A is βρ{ϕ}-compact. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets such that $B \setminus \bigcup_{\alpha \in \Delta} U_\alpha = \phi \in \{\phi\}$. Now, $A \subseteq B \Rightarrow A \setminus \bigcup_{\alpha \in \Delta} U_\alpha \subseteq B \setminus \bigcup_{\alpha \in \Delta} U_\alpha = \phi$ and this implies $A \setminus \bigcup_{\alpha \in \Delta} U_\alpha = \phi$.

Given that A is {ϕ}gβ-closed and βρ{ϕ}-compact, $\exists\Delta_o \subseteq \Delta$ finite such that $A\setminus\bigcup_{\alpha\in\Delta_o}(U_\alpha)=\phi$ and $\beta cl(A)\subseteq\bigcup_{\alpha\in\Delta_o}(U_\alpha)=\phi$. This implies $B\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha\subseteq\beta cl(A)\subseteq\bigcup_{\alpha\in\Delta_o}(U_\alpha)=\phi$. Therefore, $B\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha=\phi\in\{\phi\}$. Hence, B is βρ{ϕ}-compact.

Conversely, suppose the condition holds and B is βρ{ϕ}-compact. Let $\{U_\alpha:\alpha\in\Delta\}$ be a family of β-open sets such that $A\setminus\bigcup_{\alpha\in\Delta}U_\alpha=\phi\in\{\phi\}$. This implies $\beta cl(A)\subseteq\bigcup_{\alpha\in\Delta}U_\alpha \Rightarrow \beta cl(A)\setminus\bigcup_{\alpha\in\Delta}U_\alpha=\phi$. Now, $B\subseteq\beta cl(A) \Rightarrow B\setminus\bigcup_{\alpha\in\Delta}U_\alpha=\phi$. Also, βρ{ϕ}-compact $\Rightarrow\exists\Delta_o\subseteq\Delta$ finite such that $B\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha=\phi\in\{\phi\}$. Now, $A\subseteq B\Rightarrow A\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha\subseteq B\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha=\phi\in\{\phi\}\Rightarrow A\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha=\phi\in\{\phi\}$. Hence, A is βρ{ϕ}-compact.

Lemma: 4.19 If A and B are subsets of a topological space (X, τ) such that $A\subseteq B, B\subseteq\beta cl(A)$ and A is gβ-closed, then A is β-compact if and only if B is β-compact.

Theorem 4.20 Let $f:(X,\tau,I)\to(Y,\delta)$ be β-irresolute and surjective. If (X, τ, I) is βρI-compact and $J=\{B\subseteq Y: f^{-1}(B)\in I\}$, then (Y, δ, J) is βρI-compact.

Proof Let $f:(X,\tau,I)\to(Y,\delta)$ be β-irresolute and surjective, and (X, τ, I) be βρI-compact. By (a) of Lemma 2.19, J is an ideal on Y. Let $\{U_\alpha:\alpha\in\Delta\}$ be a family of β-open sets in Y such that $Y\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in J$. This implies $f^{-1}(Y\setminus\bigcup_{\alpha\in\Delta}U_\alpha)=X\setminus\bigcup_{\alpha\in\Delta}f^{-1}(U_\alpha)\in I$ where $\{f^{-1}(U_\alpha):\alpha\in\Delta\}$ is a family of β-open sets in X. The space (X, τ, I) is βρI-compact implies $\exists\Delta_o\subseteq\Delta$ finite such that $X\setminus\bigcup_{\alpha\in\Delta_o}f^{-1}(U_\alpha)\in I$. Hence, $f(X\setminus\bigcup_{\alpha\in\Delta_o}f^{-1}(U_\alpha))=Y\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha\in J$. Therefore, (Y, δ, J) is βρI-compact.

Theorem 4.21 Let $f:(X,\tau,I)\to(Y,\delta)$ be β-irresolute and bijective. If (X, τ, I) is βρI-compact and $f(I)=\{f(A):A\in I\}$, then (Y, δ, f(I)) is βρI-compact.

Proof Let $f:(X,\tau,I)\to(Y,\delta)$ be β-irresolute and bijective, and (X, τ, I) be βρI-compact. By (b) of Lemma 2.19, f (I) is an ideal on Y. Let $\{U_\alpha:\alpha\in\Delta\}$ be a family of β-open sets in Y such that $Y\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in f(I)$. This implies there exists $A\in I$ such that $Y\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in f(A)$. Hence, we have $A=f^{-1}(Y\setminus\bigcup_{\alpha\in\Delta}U_\alpha)=X\setminus\bigcup_{\alpha\in\Delta}f^{-1}(U_\alpha)\in I$. Given that ($X$, τ, I) is βρI-compact, $\exists\Delta_o\subseteq\Delta$ finite such that $X\setminus\bigcup_{\alpha\in\Delta_o}f^{-1}(U_\alpha)\in I$. Therefore, $f(X\setminus\bigcup_{\alpha\in\Delta_o}f^{-1}(U_\alpha))=Y\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha\in f(I)$. Hence, (Y, δ, f(I)) is βρI-compact.

Theorem 4.22 Let $f:(X,\tau)\to(Y,\delta,J)$ be an MB-open bijection. If (Y, δ, J) is βρJ-compact and $f^{-1}(J)=\{f^{-1}(C):C\in J\}$, then $(X,\tau,f^{-1}(J))$ is $\beta\rho f^{-1}(J)$-compact.

Proof Let $f:(X,\tau)\to(Y,\delta,J)$ be an MB-open bijection and (Y, δ, J) be βρJ-compact. By Lemma 2.20, $f^{-1}(J)$ is an ideal on X. Let $\{U_\alpha:\alpha\in\Delta\}$ be a family of β-open sets in X such that $X\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in f^{-1}(J)$. Then, there exists $Z\in J$ such that $X\setminus\bigcup_{\alpha\in\Delta}U_\alpha=f^{-1}(Z)$. Therefore, we have $f(X\setminus\bigcup_{\alpha\in\Delta}U_\alpha)=Y\setminus\bigcup_{\alpha\in\Delta}f(U_\alpha)=Z\in J$. Given that (Y, δ, J) is βρJ-compact, $\exists\Delta_o\subseteq\Delta$ finite such that $Y\setminus\bigcup_{\alpha\in\Delta_o}f(U_\alpha)\in J$. Hence, $f^{-1}(Y\setminus\bigcup_{\alpha\in\Delta_o}f(U_\alpha))=X\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha\in f^{-1}(J)$. Therefore, $(X,\tau,f^{-}(J))$ is $\beta\rho f^{-1}(J)$-compact.

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