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Countable compactness with respect to ideal via *β*-open sets

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ABSTRACT

In this paper, we study countable β-compactness and present another new class of compactness in ideal topological space via countable $\beta\rho I$-compact spaces by utilizing the notion of β-open sets. We also properties investigate some other of countable $\beta\rho I$-compactness using different types of functions.

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1. Introduction

Kuratowski (1966) and Vaidyanathaswamy (1945) studied the subject of ideals in topological spaces. Jankovic and Hamlett (1990, 1992) investigated further properties of ideal topological spaces. Newcomb (1967) was the first to introduce the idea of compactness in an ideal topological space and has been studied and by Racin (1972). Thoroughly, the concept was examined by Hamlett et al. (1991). The study of β-open sets was initiated by Abd El-Monsef et al. (1983), and the analogous notion of semi-preopen set was given exclusively by Andrijevic (1986) and additionally examined by Ganster and Andrijevic (1988).

An ideal I on a topological space (X,τ) is a nonempty collection of the subsets of X which satisfies the following conditions: (i) $A \in I$ and $B \subseteq A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given an ideal topological space (X,τ,I) and a subset A of X, a set operator $(.)^*: P(X) \rightarrow P(X)$ called a local function of A with respect to τ and I is defined to be the set $A^*(I,\tau) = \{x \in X: (A \cap U) \notin I$, for every $U \in U_x\}$, where $U_x = \{U \in \tau : x \in U\}$. The Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(I,\tau)$, called the $*$–topology, finer than τ, is defined by $Cl^*(A) = A \cup A^*(I,\tau)$. Where no confusion will arise, we write A^* for $A^*(I,\tau)$ and τ^* for $\tau^*(I,\tau)$. For an ideal topological space (X,τ,I), the collection $\{U \setminus A : U \in \tau$ and $A \in I\}$ is a base for τ^*. In this this paper, we utilize the notion of β-open sets to examine countable β-compactness. Furthermore, we introduce and study some types of compactness modulo ideal called $\beta\rho I$-compact. Throughout this paper, (X,τ) and (Y,δ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a set $A \subseteq X$, $cl(A)$, $int(A)$, and $X \setminus A$ denote, respectively, the closure, interior and the complement of A in (X,τ), and $P(A)$ denotes the power set of A.

2. Preliminaries

Definition: 2.1 *An ideal I on a topological space (X,τ) is a nonempty collection of the subsets of X such that if*

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(i) *$A \in I$ and $B \subseteq A$, then $B \in I$ (heredity).*

(ii) *$A \in I$ and $B \in I$, then $A \cup B \in I$ (finite additivity).*

Definition: 2.2 *An ideal topological space (or simply ideal space) denoted as (X,τ,I) is a topological space (X,τ) with an ideal I on X. An ideal I in a topological space (X,τ) is said to be* σ-ideal if it is countably additive *by* Jankovic and Hamlett (1992).

That is, if $\{A_\alpha : \alpha \in \Delta\}$ is any countable collection of members I, then $\bigcup_{\alpha\in\Delta} U_\alpha \in I$

Example: 2.3 *If (X,τ) is a topological space. The following are ideals on (X,τ) :*

$I = \{\phi\}$, $I_f = \{A \subseteq X : A$ is finite$\}$, $I_c = \{A \subseteq X : A$ is countable$\}$, $P(X)$, and $P(U) = \{A \subseteq U : U \subseteq X\}$ for any given subset U of X.

Definition: 2.4 *Let (X,τ) be a topological space. A subset A of X is said to be pre-semi-open (or β-open) if $A \subseteq cl(int(cl(A)))$* as defined by Andrijevic (1986). *The complement of a β-open set is called a β-closed set. The collection of all β-open sets in a topological space (X,τ) is denoted as $\beta O(X,\tau)$. The β-closure of A is the intersection over all β-closed sets containing A and is denoted by $\beta cl(A)$. The β-interior of A is the union over all β-open sets contained in A and is denoted by $\beta int(A)$.*

Remark: 2.5 *By Definition 2.4, it is clear that given any topological space (X,τ), if $A \in \tau$,* then *$A \subseteq cl(int(cl(A)))$. Thus, if $A \in \tau$,* then *$A \in \beta O(X,\tau)$. The complement of a β-open set is called a β-closed set, and since every open set is β-open, the compliment of every open set is β-closed.*

Therefore, every closed set is β-closed. Furthermore, $\beta O(X,\tau) \subseteq \beta O(X,\tau^*)$ since τ^* is finer than τ. Moreover, $\beta O(X,\tau)$ is closed under arbitrary union.

Definition: 2.6 *A subset A of an ideal space (X,τ,I) is said to β-open with respect to I (or βI-open) if there exist an open set U such that*

i) *$(U \setminus A) \in I$, and*

ii) *$(A \setminus cl(int(cl(U)))) \in I$* (Catalan et al., 2019)

Definition: 2.7 *A subset A of an ideal space (X,τ,I) is said to be generalized closed with respect to ideal I (briefly Ig-closed) if and only if $(cl(A) \setminus U) \in I$, whenever $A \subseteq U$ and $U \in \tau$* (Catalan et al., 2019) and (Jafari and Rajesh, 2011).

Definition: 2.8 *A subset A of an ideal space (X,τ,I) is said to be Igβ-closed if and only if $(\beta cl(A) \setminus U) \in I$, whenever $A \subseteq U$ and $U \in \beta O(X,\tau)$.*

Definition: 2.9 *A subset A of a space (X,τ) is said to be generalized-β-closed (gβ-closed) if $\beta cl(A) \subseteq U$ whenever $A \subseteq U, U \in \beta O(X,\tau)$.*

Definition: 2.10 *A function $f : (X,\tau) \to (Y,\delta)$ is said to be:*

a) *β*-irresolute, if $\forall U \in \beta O(Y,\delta)$, $f^{-1}(U) \in \beta O(X,\tau)$

b) *Mβ-open, if $\forall U \in \beta O(X,\tau)$, $f(U) \in \beta O(Y,\delta)$* (Mahmoud and Abd El-Monsef, 1990).

Lemma: 2.11 *If $f : (X,\tau) \to (Y,\delta)$ is β-irresolute surjection and* I *is an ideal on X, then*

a) *$J = \{A \subseteq Y : f^{-1}(A) \in I\}$ is ideal on Y.*

b) *$f(I) = \{f(A) : A \in I\}$ is ideal on Y* (Newcomb, 1967).

Lemma: 2.12 *If $f : (X,\tau) \to (Y,\delta)$ is Mβ-open injection and* J *is an ideal on Y, then $f^{-1}(J) = \{f^{-1}(A) : A \in J\}$ is ideal on X* (Mahmoud and Abd El-Monsef, 1990).

Lemma: 2.13 *If I and J are ideals on a topological space (X,τ), then the following hold:*

(a) *$I \cap J = \{A : A \in I$ and $A \in J\}$ is an ideal on X.*

(b) *$I \vee J = \{A \cup B : A \in I$ and $B \in J\}$ is an ideal on X* (Jankovic and Hamlett, 1992).

Definition: 2.14 *A topological space (X,τ) is said to be:*

a) *β-Hausdorff or (βT_2) space if given any two distinct points $x,y \in X$, there exist disjoint β-open sets U and V such that $x \in U, y \in V$* (Mahmoud and Abd El-Monsef, 1990).

b) *β-Urysohn or $(\beta T_{2\frac{1}{2}})$ space if given any two distinct points $x,y \in X$, there exist disjoint β-open sets U and V such that $x \in U, y \in V$ and $\beta cl(U) \cap \beta cl(V) = \phi$*

c) *β-normal if given any two disjoint closed sets A and B in X, there exist disjoint β-open sets U and V such that $A \subseteq U, B \subseteq V$* (Mahmoud and Abd El-Monsef, 1990).

d) *Completely β-normal if given any two disjoint sets A and B in X, there exist disjoint β-open sets U and V such that $A \subseteq U, B \subseteq V$.*

Definition: 2.15 *Let (X,τ) be a topological space. A collection $\{U_\alpha : \alpha \in \Delta\}$ of subsets of X is said to be:*

i) *a cover for X if $X \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$*

ii) *an open (resp., β-open) cover for X if $\{U_\alpha : \alpha \in \Delta\}$ is a cover for X and $\forall \alpha \in \Delta, U_\alpha \in \tau$ (resp., $U_\alpha \in \beta O(X,\tau)$)* (Mahmoud and Abd El-Monsef, 1990)

Definition: 2.16 *A topological space (X,τ) is said to be*

i) *compact (resp. countably compact) if for every open (resp. countable open) cover $\{U_\alpha : \alpha \in \Delta\}$ for X, $\exists \Delta_o \subseteq \Delta$ finite such that $X \subseteq \bigcup_{\alpha\in\Delta_o} U_\alpha$.*

ii) *β-compact (resp., countably β-compact) if for every β-open (resp., countable β-open) cover* $\{U_\alpha : \alpha \in \Delta\}$ *for X,* $\exists \Delta_o \subseteq \Delta$ *finite such that* $X \subseteq \bigcup_{\alpha\in\Delta_o} U_\alpha$. (Mahmoud and Abd El-Monsef, 1990)

Definition: 2.17 *An ideal space* (X,τ) *is said to be*

i) I-*compact if for every open cover* $\{U_\alpha : \alpha\in\Delta\}$ *for* X $\exists\Delta_0 \subseteq \Delta$ *finite such that* $X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \in I$. (Hosny, 2013) and (Gupta and Kaur, 2014)

ii) *Countably* I-*compact if for every countable open cover* $\{U_\alpha : \alpha\in\Delta\}$ *for* X $\exists\Delta_o \subseteq \Delta$ *finite such that* $X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \in I$ (Hamlet et al, 1991) and (Nasef, 2001)

iii) *βI-compact (resp., countably βI-compact) if for every β-open (resp., countable β-open) cover* $\{U_\alpha : \alpha\in\Delta\}$ *for X,* $\exists\Delta_0 \subseteq \Delta$ *finite such that* $X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \in I$. (Mahmoud and Abd El-Monsef, 1990)

Lemma: 2.18 *Every I-compact space is countably I-compact.*

Lemma: 2.19 *Every β-compact space is countably β-compact.*

Lemma: 2.20 *Every βI-compact space is countably βI-compact.*

3. Countably *β*-Compact Spaces

Theorem 3.1 *A β-closed subset of a countably β-compact space is countably β-compact.*

Proof Let (X,τ) be countably β-compact, B be β-closed in X and $\{U_\alpha : \alpha \in \Delta\}$ be a countable family of β open sets such that $B \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$. Then $(X \setminus B)$ is β-open and $X \subseteq \bigcup_{\alpha\in\Delta} U_\alpha \cup (X\setminus B)$. Since X is countably β-compact $\exists\Delta_o \subseteq \Delta$ finite such that $X \subseteq [\bigcup_{\alpha\in\Delta_o} U_\alpha \cup (X\setminus B)]$. Hence, $B \subseteq \bigcup_{\alpha\in\Delta_o} U_\alpha$. Therefore, B is countably β-compact.

Lemma: 3.2 *A β-closed subspace of a countably β-compact space is countably β-compact.*

Theorem 3.3 *If B is countably infinite and countably β-compact subset of a β-Hausdorff space X and* $x \notin B$*, then there exist β-open sets U and V such that* $x \in U, B \subseteq V$ *and* $U \cap V = \phi$

Proof Let B be countably infinite and countably β-compact subset of a β-Hausdorff space X and $x \notin B$. Now $\forall b \in B$, $x \neq b$ and since X is β-Hausdorff, there exist β-open sets U_x and V_b such that $x \in U_x, b \in V_b$ and $U_x \cap V_b = \phi$

The family $\{V_b : b \in B\}$ forms a countable β-open cover for B. Since B is countably β-compact, there exist a finite subcollection $\{V_{bi} : i = 1,2,...,n\}$ such that $B \subseteq \bigcup_{i=1}^{n} V_{b_i}$. Let $V = \bigcup_{i=1}^{n} V_{b_i}$, then V is β-open and $B \subseteq V$. For each β-open set V_{bi} : $i = 1,2,...,n$ there exist a corresponding β-open set U_{xi} such that $x \in U_{xi}$ and $V_{bi} \cap U_{xi} = \phi$. Let $U = \beta\,\mathrm{int}(\bigcap_{i=1}^{n} U_{x_i})$, then U is β-open, $x \in U$ and $U \cap V = \phi$.

Theorem 3.4 *A countably infinite and countably β-compact subset of a β-Hausdorff space is β-closed.*

Proof Let X be β-Hausdorff and $B \subseteq X$ be countably infinite and countably β-compact. Let $x \notin B$, then by Theorem 3.3, there exist β-open sets U and V such that $x \in U, B \subseteq V$ and $U \cap V = \phi$. This implies $\forall x \notin B$ there exist β-open set U_x such that $x \in U_x$ and $U_x \cap B = \phi$. Let $H = \bigcup_{x\notin B} U_x$, then H is β-open and $(X \setminus H) = B$. Hence, B is β-closed.

Remark: 3.5 *It should be noted that in Theorem 3.3 and Theorem 3.4 if B is finite, then the results are obvious.*

Theorem 3.6 *A space* (X,τ) *is countably β-compact iff every countable family* $\{U_\alpha : \alpha \in \Delta\}$ *of β-closed sets in X having the finite intersection property has a nonempty intersection.*

Proof Let (X,τ) be countably β-compact and $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-closed sets in X having the finite intersection property. Suppose $\bigcap_{\alpha\in\Delta} U_\alpha = \phi$, then $\bigcup_{\alpha\in\Delta}(X\setminus U_\alpha) = X \setminus \bigcap_{\alpha\in\Delta} U_\alpha = X\setminus\phi = X$. This implies the collection $\{X \setminus U_\alpha : \alpha\in\Delta\}$ is a countable β-open cover for X. Hence, there is a finite subcover say $\{(X \setminus U_{\alpha i}) : i = 1,2,...,n\}$ such that $X \subseteq \bigcup_{i=1}^{n}(X\setminus U_{\alpha_i}) = X \setminus \bigcap_{i=1}^{n} U_{\alpha_i}$. This implies $\bigcap_{i=1}^{n} U_{\alpha_i} = \phi$, this is a contradiction, i.e., $\{U_\alpha : \alpha \in \Delta\}$ has the finite intersection property. Hence, $\bigcap_{\alpha\in\Delta} U_\alpha \neq \phi$.

Conversely, let the condition holds and $\{U_\alpha : \alpha\in\Delta\}$ be a countable β-open cover of X. Then, $\{(X \setminus U_\alpha) : \alpha\in\Delta\}$ is a countable family of β-closed subsets of X such that $\bigcap_{\alpha\in\Delta}(X\setminus U) = X \setminus \bigcup_{\alpha\in\Delta} U_\alpha = X\setminus X = \phi$. By our hypothesis the collection $\{(X \setminus U_\alpha) : \alpha\in\Delta\}$ does not have the finite intersection property.

Therefore, there is a finite subcollection such that $\bigcap_{i=1}^{n}(X\setminus U_{\alpha_i}) = \phi$. Now, $\bigcap_{i=1}^{n}(X\setminus U_{\alpha_i}) = X \setminus \bigcup_{i=1}^{n} U_{\alpha_i} = \phi \Rightarrow X \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. Hence, $\{U_{\alpha i} : i = 1,2,...,n\}$ is a finite subcover of $\{U_\alpha : \alpha\in\Delta\}$ for X. Therefore, (X,τ) is countably β-compact.

Theorem 3.7 *If* $f : X \to Y$ *is β-irresolute and X is countably β-compact, then Y is countably β-compact.*

Proof Let $f : X \to Y$ be β-irresolute, X be countably β-compact, and $\{U_\alpha : \alpha \in\Delta\}$ be a countable β-open cover for Y. This implies $Y \setminus \bigcup_{\alpha\in\Delta} U_\alpha = \phi$ and $f^{-1}(Y\setminus\bigcup_{\alpha\in\Delta} U_\alpha) = X \setminus f^{-1}(\bigcup_{\alpha\in\Delta} U_\alpha) = \phi$. The collection $\{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is a family of β-open sets in X and $X \setminus \bigcup_{\alpha\in\Delta} f^{-1}(U_\alpha) = \phi \Rightarrow X \subseteq \bigcup_{\alpha\in\Delta} f^{-1}(U_\alpha)$. Since X is countably β-compact, $\exists\Delta_o \subseteq \Delta$ finite such that $X \subseteq \bigcup f^{-1}(U_\alpha)$. This implies $X \setminus \bigcup_{\alpha\in\Delta_o} f^{-1}(U_\alpha) = \phi$ and $f[X \setminus \bigcup_{\alpha\in\Delta_o} f^{-1}(U_\alpha)] = Y \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha = \phi$. Hence, $Y \subseteq \bigcup_{\alpha\in\Delta_o} U_\alpha = \phi$. Therefore, Y is countably β-compact.

Theorem 3.8 *If $f: X \to Y$ is bijective, $M\beta$-open and Y is countably β-compact, then X is countably β-compact.*

Proof Let $f: X \to Y$ be bijective, $M\beta$-open, Y be countably β-compact and $\{U_\alpha : \alpha \in \Delta\}$ be a countable β-open cover for X. Now, $X \subseteq \bigcup_{\alpha\in\Delta} U_\alpha \Rightarrow X \setminus \bigcup_{\alpha\in\Delta} U_\alpha = \phi$ $\Rightarrow f(X \setminus \bigcup_{\alpha\in\Delta} U_\alpha) = Y \setminus \bigcup_{\alpha\in\Delta} f(U_\alpha) = \phi \Rightarrow \{f(U_\alpha) : \alpha \in \Delta\}$ is a countable -open cover for Y. By the hypothesis, $\exists \Delta_o \subseteq \Delta$ finite such that $Y \subseteq f(\bigcup_{\alpha\in\Delta_o} U_\alpha)$. Hence, $Y \setminus f(\bigcup_{\alpha\in\Delta_o} U_\alpha) = \phi$. This implies $f^{-1}(Y \setminus f(\bigcup_{\alpha\in\Delta_o} U_\alpha)) = X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha = \phi$. Therefore, X is countably β-compact.

4. Countably $\beta\rho I$-Compact Spaces

Definition: 4.1: *A subset A of an ideal space (X,τ,I) is said to be ρI-compact if for every family $\{U_\alpha : \alpha \in \Delta\}$ of open sets, if $(A \setminus \bigcup_{\alpha\in\Delta} U_\alpha) \in I$, then there exists $\Delta_0 \subseteq \Delta$ finite such that $(A \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in I$. The space (X,τ,I) is said to be ρI-compact if X is ρI-compact* (Pachon, 2016).

Definition: 4.2 *A subset A of an ideal space (X,τ,I) is said to be countably ρI-compact if for every countable family $\{U_\alpha : \alpha \in \Delta\}$ of open sets; if $(A \setminus \bigcup U_\alpha) \in I$, then there exists $\Delta_0 \subseteq \Delta$ finite such that $(A \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in I$. The space (X,τ,I) is said to be countably ρI-compact if X is countably ρI-compact.*

Definition: 4.3 *A subset A of an ideal space (X,τ,I) is said to be $\beta\rho I$-compact (resp. countably $\beta\rho I$-compact) if for every (resp. countable)family $\{U_\alpha : \alpha \in \Delta\}$ of β-open sets , if $(A \setminus \bigcup_{\alpha\in\Delta} U_\alpha) \in I$ then there exists $\exists \Delta_0 \subseteq \Delta$ finite such that $(A \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha) \in I$. The space (X,τ,I) is said to be $\beta\rho I$-compact (resp. countably $\beta\rho I$-compact) if X is $\beta\rho I$-compact (resp. countably $\beta\rho I$-compact).*

Lemma: 4.4 *If $(X,\tau,\{\phi\})$ is countably $\beta\rho\{\phi\}$-compact and B is β-closed in X, then B is countably $\beta\rho\{\phi\}$compact space. (Theorem 3.1)*

Theorem 4.5 *A topological space (X,τ) is countably β-compact iff $(X,\tau,\{\phi\})$ is $\beta\rho\{\phi\}$-compact.*

Proof *A* space (X,τ) is countably β-compact $\Rightarrow$ given any collection $\{U_\alpha : \alpha \in \Delta\}$ of β-open sets such that $X \setminus \bigcup_{\alpha\in\Delta} U_\alpha = \phi$, $\exists \Delta_o \subseteq \Delta$ finite such that $X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha = \phi \in \{\phi\}$. *This implies* $(X,\tau,\{\phi\})$ is countably $\beta\rho\{\phi\}$-compact.

Conversely, let $(X,\tau,\{\phi\})$ be countably $\beta\rho\{\phi\}$-compact and $\{U_\alpha : \alpha \in \Delta\}$ be any β-open cover for X. This implies $X \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ and $X \setminus \bigcup_{\alpha\in\Delta} U_\alpha = \phi \in \{\phi\}$. Since $(X,\tau,\{\phi\})$ is countably $\beta\rho\{\phi\}$-compact, $\exists \Delta_o \subseteq \Delta$ finite, such that $X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha = \phi \in \{\phi\}$. *This implies* $X \subseteq \bigcup_{\alpha\in\Delta_o} U_\alpha$. Hence, (X,τ) is countably β-compact.

Lemma: 4.6 *If (X,τ,I) is countably $\beta\rho I$-compact for every ideal I, then (X,τ) is countably β-compact.*

Theorem 4.7 *If (X,τ,I) is countably $\beta\rho I$-compact , then (X,τ,I) is countably ρI-compact.*

Proof Let (X,τ,I) be countably $\beta\rho I$-compact and $\{U_\alpha : \alpha \in \Delta\}$ be a countable family of open sets in X, such that $X \setminus \bigcup_{\alpha\in\Delta} U_\alpha \in I$. By Remark 2.5, $U_\alpha \in \beta O(X,\tau), \forall \alpha \in \Delta$. By this hypothesis $\exists \Delta_0 \subseteq \Delta$ finite such that $X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \in I$. Hence, (X,τ,I) is countably ρI-compact.

Theorem 4.8 *An ideal space (X,τ,I) is countably $\beta\rho I$-compact iff (X,τ^*,I) is countably $\beta\rho I$-compact.*

Proof Let (X,τ,I) be countably $\beta\rho I$-compact and $\{U_\alpha : \alpha \in \Delta\}$ be a countable collection of β-open sets in (X,τ^*,I) such that $X \setminus \bigcup_{\alpha\in\Delta} U_\alpha \in I$. Now, $U_\alpha = V_\alpha \setminus W_\alpha$ such that $V_\alpha \in \beta O(X,\tau)$, $W_\alpha \in I, \forall \alpha \in \Delta$. Therefore, $\{V_\alpha : \alpha \in \Delta\}$ is a countable collection of β-open sets in (X,τ,I) and $\bigcup_{\alpha\in\Delta} U_\alpha \subseteq \bigcup_{\alpha\in\Delta} V_\alpha$. Therefore, $X \setminus \bigcup_{\alpha\in\Delta} V_\alpha \subseteq X \setminus \bigcup_{\alpha\in\Delta} U_\alpha \in I \Rightarrow X \setminus \bigcup_{\alpha\in\Delta} V_\alpha \in I$. By this hypothesis $\exists \Delta_0 \subseteq \Delta$ finite such that $X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \in I$.

Now, $X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \subseteq [(X \setminus \bigcup_{\alpha\in\Delta_o} V_\alpha) \cup (\bigcup_{\alpha\in\Delta_o} W_\alpha)] \in I \Rightarrow X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \in I \Rightarrow$ (X,τ^*,I) is countably $\beta\rho I$-compact.

Conversely, suppose (X,τ^*,I) is countably $\beta\rho I$-compact and $\{U_\alpha : \alpha \in \Delta\}$ is a countable collection of β-open sets in (X,τ,I) such that $X \setminus \bigcup_{\alpha\in\Delta} U_\alpha \in I$. By Remark 2.5 and the hypothesis $\{U_\alpha : \alpha \in \Delta\}$ is a countable collection of β-open sets in (X,τ^*,I) and $\exists \Delta_0 \subseteq \Delta$ finite such that $X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \in I$.

Hence, (X,τ,I) is countably $\beta\rho I$-compact.

Theorem 4.9 *An ideal space (X,τ,I) is countably $\beta\rho I$-compact iff for any countable family $\{V_\alpha : \alpha \in \Delta\}$ of β-closed sets in X; if $\bigcap_{\alpha\in\Delta} V_\alpha \in I$, then there exists $\Delta_0 \subseteq \Delta$ finite such that $\bigcap_{\alpha\in\Delta_o} V_\alpha \in I$.*

Proof Let (X,τ,I) be countably $\beta\rho I$-compact and $\{V_\alpha : \alpha \in \Delta\}$ be a countable family of β-closed sets in X such that $\bigcap_{\alpha\in\Delta} V_\alpha \in I$. Now if $U_\alpha = (X \setminus V_\alpha), \forall \alpha \in \Delta$, then $\{U_\alpha : \alpha \in \Delta\}$ is a countable family of β-open sets in X, and we have $\bigcap_{\alpha\in\Delta} V_\alpha = \bigcap_{\alpha\in\Delta} (X \setminus U_\alpha) = X \setminus \bigcup_{\alpha\in\Delta} U_\alpha \in I$. By the hypothesis $\exists \Delta_o \subseteq \Delta$ finite such that $X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \in I$.

Hence, $X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha = \bigcap_{\alpha\in\Delta_o} (X \setminus U_\alpha) = \bigcap_{\alpha\in\Delta_o} V_\alpha \in I$

Conversely, suppose the condition holds and let $\{U_\alpha : \alpha \in \Delta\}$ be a countable family of β-open sets in X such that $X \setminus \bigcup_{\alpha\in\Delta} U_\alpha \in I$. Then $\{(X \setminus U_\alpha) : \alpha \in \Delta\}$ is a family of β-closed sets and $X \setminus \bigcup_{\alpha\in\Delta} U_\alpha = \bigcap_{\alpha\in\Delta} (X \setminus U_\alpha) \in I$. By our hypothesis $\exists \Delta_o \subseteq \Delta$ finite such that $\bigcap_{\alpha\in\Delta_o} (X \setminus U_\alpha) \in I$. Therefore, $\exists \Delta_o$

$\subseteq\Delta$ finite such that $\bigcap_{\alpha\in\Delta_o}(X\setminus U_\alpha)=X\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha\in I$. Hence, (X,τ,I) is countably $\beta\rho I$-compact.

Corollary: 4.10 (X,τ^*,I) *is countably* $\beta\rho I$*-compact iff for any countable family* $\{V_\alpha : \alpha\in\Delta\}$ *of* β*-closed sets in* (X,τ,I)*, if* $\bigcap_{\alpha\in\Delta}V_\alpha\in I$ *then there exists* $\exists\Delta_0\subseteq\Delta$ *finite such that* $\bigcap_{\alpha\in\Delta_o}V_\alpha\in I$.

Theorem 4.11 *Let* I *and* J *be ideals in* (X,τ) *and* $K = I\cap J$*. If* (X,τ,I) *and* (X,τ,J) *are countably* $\beta\rho I$*-compact, then* (X,τ,K) *is* $\beta\rho I$*-compact*

Proof Suppose the condition holds. Let $\{U_\alpha : \alpha\in\Delta\}$ be any countable family of β-open sets in X such that $X\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in K$. This implies $X\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in I$ and $X\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in J$. Given that (X,τ,I) and (X,τ,J) are countably $\beta\rho I$-compact $\Rightarrow\exists\Delta_1$ and Δ_2 finite such that $X\setminus\bigcup_{\alpha\in\Delta_1}U_\alpha\in I$ and $X\setminus\bigcup_{\alpha\in\Delta_2}U_\alpha\in J$. Taking $\Delta_3 = \Delta_1\cap\Delta_2$, then Δ_3 is finite and $X\setminus\bigcup_{\alpha\in\Delta_3}U_\alpha\in I$ and $X\setminus\bigcup_{\alpha\in\Delta_3}U_\alpha\in J$. Therefore, $X\setminus\bigcup_{\alpha\in\Delta_3}U_\alpha\in(I\cap J)\Rightarrow X\setminus\bigcup_{\alpha\in\Delta_3}U_\alpha\in K$.

Hence,(X,τ,K) is countably $\beta\rho I$-compact.

Theorem 4.12 *If A and B are countably* $\beta\rho I$*-compact subsets of an ideal space* (X,τ,I)*, then* $A\cup B$ *is countably* $\beta\rho I$*-compact.*

Proof Let A and B be countably $\beta\rho I$-compact subsets of an ideal space (X,τ,I) and $\{U_\alpha : \alpha\in\Delta\}$ be a countable family of β-open sets such that $(A\cup B)\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in I$.

Now $(A\cup B)\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in I.\Rightarrow A\setminus\bigcup_{\alpha\in\Delta}U_\alpha\subseteq(A\cup B)\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in I.$

Similarly $B\setminus\bigcup_{\alpha\in\Delta}U_\alpha\subseteq(A\cup B)\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in I.$ Since A and B are countably $\beta\rho I$-compact, then $\exists\Delta_1\subseteq\Delta$ and $\Delta_2\subseteq\Delta$ finite such that $A\setminus\bigcup_{\alpha\in\Delta_1}U_\alpha\in I$ and $B\setminus\bigcup_{\alpha\in\Delta_2}U_\alpha\in I$. Taking $\Delta_3 = \Delta_1\cup\Delta_2$, then Δ_3 is finite and $(A\cup B)\setminus\bigcup_{\alpha\in\Delta_3}U_\alpha\in I$. Hence, $A\cup B$) is countably $\beta\rho I$-compact.

Lemma: 4.13 *Finite union of countably* $\beta\rho I$*-compact sets is countably* $\beta\rho I$*-compact.*

Theorem 4.14 *Let A be a countably* $\beta\rho I$*-compact subset of an ideal space* (X,τ,I)*. If B is a* β*-open set contained in A, then* $(A\setminus B)$ *is countably* $\beta\rho I$*-compact.*

Proof Let A be a countably $\beta\rho I$-compact subset of an ideal space (X,τ,I) and B be a β-open set such that $B\subseteq A$. Let $\{U_\alpha : \alpha\in\Delta\}$ be a countable family of β-open sets in X such that $(A\setminus B)\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in I$.

This implies $A\setminus\bigcup_{\alpha\in\Delta}(B\cup U_\alpha)\in I$. Given A is countably $\beta\rho I$-compact and B is a β-open set, $\exists\Delta_o\subseteq\Delta$ finite such that $A\setminus\bigcup_{\alpha\in\Delta_o}(B\cup U_\alpha)\in I$. This implies $(A\setminus B)\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha\in I$. Hence, $(A\setminus B)$ is countably $\beta\rho I$-compact.

Theorem 4.15 *Let* (X,τ,I) *be an ideal space and* $A\subseteq X$*. If for every countable collection* $\{U_\alpha : \alpha\in\Delta\}$ *of* β*-open sets, if* $(A\setminus\bigcup_{\alpha\in\Delta}U_\alpha)\in I$*, there exists a countably* $\beta\rho I$*-compact set B containing A such that* $(B\setminus\bigcup_{\alpha\in\Delta}U_\alpha)\in I$*, then A is countably* $\beta\rho I$*-compact.*

Proof Suppose the condition holds. Let $A\subseteq X$,$\{U_\alpha : \alpha\in\Delta\}$ be a countable collection of β-open sets such that $(A\setminus\bigcup_{\alpha\in\Delta}U_\alpha)\in I$ and B be countably $\beta\rho I$-compact such that $A\subseteq B$ and $(B\setminus\bigcup_{\alpha\in\Delta}U_\alpha)\in I$. Now B is countably $\beta\rho I$- compact $\exists\Delta_o\subseteq\Delta$finite such that $(B\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha)\in I$. And, $A\subseteq B$ implies $(A\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha)\in I$. This implies A is countably $\beta\rho I$-compact.

Theorem 4.16 *If* $(X,\tau,\{\phi\})$ *is countably* $\beta\rho\{\phi\}$*-compact, then every* $\{\phi\}g\beta$*-closed subset of* $(X,\tau,\{\phi\})$ *is countably* β*-compact.*

Proof Let A be a $\{\phi\}g\beta$-closed subset of a countably $\beta\rho\{\phi\}$-compact space $(X,\tau,\{\phi\})$ and $\{U_\alpha : \alpha\in\Delta\}$ be a countable collection β-open sets such that $A\subseteq\bigcup_{\alpha\in\Delta}U_\alpha$.

Now A is $\{\phi\}g\beta$-closed implies $\beta cl(A)\subseteq\bigcup_{\alpha\in\Delta}U_\alpha$. Therefore, $X=(X\setminus\beta cl(A))\cup\bigcup_{\alpha\in\Delta}U_\alpha$, this implies $X\setminus[(\beta cl(A))\cup(\bigcup_{\alpha\in\Delta}U_\alpha)]=\phi\in\{\phi\}$. Since $(X\setminus\beta cl(A))$ is β-open and (X,τ,I) is countably $\beta\rho I$-compact, there exist $\exists\Delta_o\subseteq\Delta$ finite such that $X\setminus[(\beta cl(A))\cup(\bigcup_{\alpha\in\Delta_o}U_\alpha)]=\phi\in\{\phi\}$.

However, $X\setminus[(\beta cl(A))\cup(\bigcup_{\alpha\in\Delta_o}U_\alpha)]=\beta cl(A)\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha$. Hence, $\beta cl(A)\subseteq\bigcup_{\alpha\in\Delta_o}U_\alpha$. Therefore, A is countably β- compact.

Theorem 4.17 *If A and B are subsets of an ideal space* $(X,\tau,\{\phi\})$ *such that* $A\subseteq B$*,*$B\subseteq\beta cl(A)$*, and A is* $\{\phi\}g\beta$*closed, then A is countably* $\beta\rho\{\phi\}$*-compact iff B is countably* $\beta\rho\{\phi\}$*-compact.*

Proof Suppose the condition holds and A is countably $\beta\rho\{\phi\}$-compact. Let $\{U_\alpha : \alpha\in\Delta\}$ be a countable family of β-open sets such that $B\setminus\bigcup_{\alpha\in\Delta}U_\alpha=\phi\in\{\phi\}$.

Now, $A\subseteq B\Rightarrow A\setminus\bigcup_{\alpha\in\Delta}U_\alpha\subseteq B\setminus\bigcup_{\alpha\in\Delta}U_\alpha=\phi\Rightarrow A\setminus\bigcup_{\alpha\in\Delta}U_\alpha=\phi$. Given that A is $\{\phi\}g\beta$-closed and countably $\beta\rho\{\phi\}$-compact $\exists\Delta_o\subseteq\Delta$ finite such that $A\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha=\phi$ and $\beta cl(A)\subseteq\bigcup_{\alpha\in\Delta_o}U_\alpha$.

Thisimplies $B\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha\subseteq\beta cl(A)\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha=\phi\Rightarrow B\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha=\phi\in\{\phi\}$. Hence, B is countably $\beta\rho\{\phi\}$-compact.

Conversely, suppose the condition holds. Let B be countably $\beta\rho\{\phi\}$-compact and $\{U_\alpha : \alpha\in\Delta\}$ be a countable family of β-open sets such that $A\setminus\bigcup_{\alpha\in\Delta}U_\alpha=\phi\in\{\phi\}$. Now A is $\{\phi\}g\beta$-closed implies $\beta cl(A)\subseteq\bigcup_{\alpha\in\Delta}U_\alpha$, giving $\beta cl(A)\setminus\bigcup_{\alpha\in\Delta}U_\alpha=\phi$.

Also, $B\subseteq\beta cl(A)\Rightarrow B\setminus\bigcup_{\alpha\in\Delta}U_\alpha\subseteq\beta cl(A)\setminus\bigcup_{\alpha\in\Delta}U_\alpha=\phi\Rightarrow B\setminus\bigcup_{\alpha\in\Delta}U_\alpha=\phi$. B is countably $\beta\rho\{\phi\}$-compact $\Rightarrow\exists\Delta_o\subseteq\Delta$ finite such that $B\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha=\phi\in\{\phi\}$. Since A is contained in B, we have $A\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha\subseteq B\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha=\phi\in\{\phi\}\Rightarrow A\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha=\phi\in\{\phi\}$.

Hence, A is countably $\beta\rho\{\phi\}$-compact,

Lemma: 4.18 *If A and B are subsets of a topological space* (X,τ) *such that* $A\subseteq B, B\subseteq\beta cl(A)$, *and A is gβ-closed, then A is countably β-compact iff B is countably β-compact.*

Theorem 4.19 *Let* $f:(X,\tau,I)\to(Y,\delta)$ *be β-irresolute, surjective, and* $J=\{B\subseteq Y: f^{-1}(B)\in I\}$. *If* (X,τ,I) *is countably βρI-compact, then* (Y,δ,J) *is countably βρJ-compact.*

Proof Let $f: X\to Y$ be β-irresolute, surjective, and (X,τ,I) be countably $\beta\rho I$-compact. By (a) of Lemma 2.11, J is an ideal on Y. Let $\{U_\alpha:\alpha\in\Delta\}$ be a countable family of β-open sets in Y such that $Y\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in J$. This implies $f^{-1}(Y\setminus\bigcup_{\alpha\in\Delta}U_\alpha)\in I$ and $\{f^{-1}(U_\alpha):\alpha\in\Delta\}$ is a countable family of β-open sets in X. Now, $f^{-1}(Y\setminus\bigcup_{\alpha\in\Delta}U_\alpha)=X\setminus\bigcup_{\alpha\in\Delta}f^{-1}(U_\alpha)\in I$ and (X,τ,I) is countably $\beta\rho I$-compact implies $\exists\Delta_o\subseteq\Delta$ finite such that $X\setminus\bigcup_{\alpha\in\Delta_o}f^{-1}(U_\alpha)\in I$.

Hence, $f(X\setminus\bigcup_{\alpha\in\Delta_o}f^{-1}(U_\alpha))=Y\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha\in J$. Therefore, (Y,δ,J) is countably $\beta\rho$J-compact.

Theorem 4.20 *Let* $f:(X,\tau,I)\to(Y,\delta)$ *be β-irresolute, bijective, and* $f(I)=\{f(A): A\in I\}$. *If* (X,τ,I) *is countably βρI-compact, then* $(Y,\delta,f(I))$ *is countably βρ f(I)-compact.*

Proof Let $f: X\to Y$ be bijective, β-irresolute, and (X,τ,I) be countably $\beta\rho I$-compact. By (b) of Lemma 2.11, $f(I)$ is an ideal on Y. Let $\{U_\alpha:\alpha\in\Delta\}$ be a countable family of β-open sets in Y such that $Y\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in f(I)$. This implies $\exists A\in I$ such that $Y\setminus\bigcup_{\alpha\in\Delta}U_\alpha=f(A)$.

Hence, $A=f^{-1}(Y\setminus\bigcup_{\alpha\in\Delta}U_\alpha)=X\setminus\bigcup_{\alpha\in\Delta}f^{-1}(U_\alpha)\in I$. Given that (X,τ,I) is countably $\beta\rho I$- compact $\exists\Delta_0\subseteq\Delta$ finite such that $X\setminus\bigcup_{\alpha\in\Delta_o}f^{-1}(U_\alpha)\in I$. Therefore, $f(X\setminus\bigcup_{\alpha\in\Delta_o}f^{-1}(U_\alpha))=Y\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha\in f(I)$. Hence, $(Y,\delta,f(I))$ is countably $\beta\rho$ f(I)-compact.

Theorem 4.21 *Let* $f:(X,\tau)\to(Y,\delta,J)$ *be Mβ-open bijection and* $f^{-1}(J)=\{f^{-1}(C): C\in J\}$. *If* (Y,δ,J) *is countably βρJ-compact, then* $(X,\tau,f^{-1}(J))$ *is countably βρ* $f^{-1}(J)$*-compact.*

Proof Let $f:(X,\tau)\to(Y,\delta,J)$ be an $M\beta$-open bijection and (Y,δ,J) be countably $\beta\rho$J-compact. By Lemma 2.12, $f^{-1}(J)$ is an ideal on X. Let $\{U_\alpha:\alpha\in\Delta\}$ be a countable family of β-open sets in X such that $X\setminus\bigcup_{\alpha\in\Delta}U_\alpha\in f^{-1}(J)$. Then, there exists $Z\in J$ such that $X\setminus\bigcup_{\alpha\in\Delta}U_\alpha=f^{-1}(Z)$.

Therefore we have $f(X\setminus\bigcup_{\alpha\in\Delta}U_\alpha)=Y\setminus\bigcup_{\alpha\in\Delta}f(U_\alpha)=Z\in J$.. Given that (Y,δ,J) is countably $\beta\rho$J-compact, $\exists\Delta_0\subseteq\Delta$ finite such that $Y\setminus\bigcup_{\alpha\in\Delta_o}f(U_\alpha)\in J$.

Therefore, $f^{-1}(Y\setminus\bigcup_{\alpha\in\Delta_o}f(U_\alpha))=X\setminus\bigcup_{\alpha\in\Delta_o}U_\alpha\in f^{-1}(J)$. Hence, $(X,\tau,f^{-1}(J))$ is countably $\beta\rho$ $f^{-1}(J)$-compact.

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