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Solution of third-order linear homogeneous fuzzy ordinary differential equations by Laplace transform method using the generalized Hukuhara differentiability concept

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ABSTRACT

This work considers the solution of third-order linear homogeneous fuzzy ordinary differential equations (FODEs) based on the fuzzy Laplace transform method under the generalized Hukuhara differentiability concept. The results of the third-order linear homogeneous is established and tested on some existing biomathematical models. C^{++} code is used to test the accuracy of the method. Finally, a recommendation for further studies is made on the establishment of results for odd- and even-order FODEs.

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1. Introduction

Fuzzy linear homogeneous differential equations and initial value problems were extensively treated by Congxin and Shiji (1998), Buckley and Feuring (2002), and Bede et al. (2007). See also Sankar and Tapan (2015) for second order differential equation in fuzzy environment.

Third-order linear fuzzy differential equations (FDEs) are one of the simplest FDEs which may appear in many applications (Bede and Gal, 2007). Strongly generalized differentiability was introduced in Bede et al.'s (2007) study and deliberated in Khastan et al.'s (2005) study. The generalized differentiability concept for the solution of third-order linear FDEs in some special cases is presented in Khastan et al.'s (2005) work.

Third-order linear and nonlinear fuzzy ordinary differential equations (FODEs) appeared in many applications that can be solved by approaches such as Adomian decomposition, variation of parameters, Laplace transform, Lagrange multiplier, embedding, differential transformation methods, Zadeh's extension principle, etc. The solution of third-order FODEs which satisfies fuzzy initial or boundary conditions has occurred in many practical problems and most importantly because sometimes it is impossible to find their analytic solutions. Solving FDEs using analytical methods mentioned above often helps to better understand a physical problem and also help to solve difficult nonlinear problems. In this study, the Laplace transform method is considered because it is the most accurate method for solving both linear and nonlinear ordinary differential equations in fuzzy environments among other analytical methods (Bede and Gal, 2004).

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This study considers the solution of third-order linear homogeneous FODEs using the fuzzy Laplace transform method (FLTM), which is treated under the generalized Hukuhara differentiability concept.

2. Methodology

The concepts of gH-differentiability and FLTM are used for solving third-order linear homogeneous. The same processes can also be applied in linear nonhomogeneous cases as well.

2.1. Concept of gH-differentiability for third-order FODEs

The third-order gH -derivative of a fuzzy valued function $f:[a,b] \to \Re_F$ at t_0 is defined as follows:

$$f'''(t_0) = \lim_{h\to 0} \frac{f''(t_0 + h) - {}_g H f''(t_0)}{h}.$$

If $f'''(t_0) \in \Re_F$, then we say that $f''(t_0)$ is gH -differentiable at t_0.

1. $f''(t_0)$ is (i) gH -differentiable at t_0 if $f'''(t_0,\alpha)$ $= \left(f_1''(t_0,\alpha), f_2''(t_0,\alpha)\right)$.
2. f is (i) gH -differentiable on (a,b) if $f'''(t_0,\alpha)$ $= \left(f_2''(t_0,\alpha), f_1''(t_0,\alpha)\right)$.
3. f is (ii) gH -differentiable on (a,b) for all $\alpha \in [0,1]$.
4. $f''(t_0)$ is (ii) gH -differentiable at t_0 if $f'''(t_0,\alpha)$ $= \left(f_2''(t_0,\alpha), f_1''(t_0,\alpha)\right)$.
5. f is (i) gH differentiable on (a,b) if $f'''(t_0,\alpha)$ $= \left(f_1''(t_0,\alpha), f_2''(t_0,\alpha)\right)$
6. f is (ii) gH -differentiable on (a,b) for all $\alpha \in [0,1]$.

2.2. Laplace transform method for third-order FODEs

Suppose:

i. The derivative D takes a differentiable function f (defined on some intervals (a,b)) and assigns to it a new function $Df = f'$.

ii. The integral I takes a continuous function f (defined on some intervals $[a,b]$) and assigns to it a new function

$$If(x) = \int_a^x f(t)dt.$$

iii. The multiplication operator M_φ, which multiplies any given function f on the interval $[a,b]$ by a fixed function φ on $[a,b]$, is a transform:

$$M_\varphi f(x) = \varphi(x)\cdot f(x).$$

The Laplace transform of the derivatives is

$$L[y'] = \int_0^\infty e^{-px} y'(x)dx$$

and in general,

$$L[f^n(t)] = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0).$$

3. Results and Discussion

3.1. Results

Consider the following equation:

$$y''' = k\,y \tag{1}$$

With the following initial conditions:

$$\begin{aligned} &y(0)=x_0 \\ &y'(0)=y_0 \\ &y''(0)=z_0 \end{aligned} \tag{2}$$

$$L\left[y'''\right] = k\,L\left[y\right] \tag{3}$$

Applying lemma (1) and (2), and taking the Laplace transform of Equation (3) and simplifying it, we have

$$kL\left[f(x)\right] = s^3 L\left[f(t,\alpha)\right] (4)$$

Now, we have the following alternatives for solving Equation (4).

Case 1, $a \ge 0$ and $b \ge 0$; case 2, $a \ge 0$ and $b < 0$; case 3, $a < 0$ and $b \ge 0$; and case 4, $a < 0$ and $b < 0$.

Suppose that the fuzzy linear function f is given as follows:

$$ay''(t) + by'''(t)\ (5)$$

Taking the Laplace transform of Equation (5) and simplify we have

$$f(t,\alpha) = s^2(a+bs)L\left[y(t,\alpha)\right] - s(a+s)x_0(\alpha) - (a+s)y_0(\alpha) - z_0(\alpha) \tag{6}$$

Now, equating and simplify Equations (5) and (6) we have

$$\frac{1}{k}\left[s^3L\left[f(t,\alpha)\right]-s^2x_0(\alpha)-sy_0(\alpha)-z_0(\alpha)\right]= s^2\left[a+bs\right]L\left[y(t,\alpha)\right]-s(a+b)x_0(\alpha)-(a+s)y_0(\alpha)-z_0(\alpha) \quad (7)$$

Solving Equation (7) we have

$$s^3L\left[y(t,\alpha)\right]-s^2x_0(\alpha)-sy_0(\alpha)-z_0(\alpha)=ks^2\left[a+bs\right]L\left[y(t,\alpha)\right]-ks(a+b)x_0(\alpha) \quad (8)$$

Now, applying the cases to Equation (8) we have

1. If $a \geq 0$ and $b \geq 0$

Case 1, $a \geq 0$ and $b \geq 0$; case 2, if $a \geq 0$ and $b < 0$; case 3, if $a < 0$ and $b \geq 0$; case 4, if $a < 0$ and $b < 0$

$$s^3L\left[\underline{y}(t,\alpha)\right]-s^2\underline{x}_0(\alpha)-s\underline{y}_0(\alpha)-\underline{z}_0(\alpha)= ks^2\left[a+bs\right]L\left[\underline{y}(t,\alpha)\right]-ks(a+b)\underline{x}_0(\alpha)-(a+s)\underline{y}_0(\alpha)-\underline{z}_0(\alpha) \quad (9)$$

$$s^3L\left[\overline{y}(t,\alpha)\right]-s^2\overline{x}_0(\alpha)-s\overline{y}_0(\alpha)-\overline{z}_0(\alpha)=ks^2\left[a+bs\right]L\left[\overline{y}(t,\alpha)\right]-ks(a+b)\overline{x}_0(\alpha) \quad (10)$$

Solving Equation (10) we have

$$L\left[\underline{y}(t,\alpha)\right]=\frac{\left[s^2+ks(a+bs)\right]\underline{x}_0(\alpha)+\left[s-(a+s)\right]\underline{y}_0(\alpha)}{s^3-ks(a+bs)} \quad (11)$$

Taking the inverse Laplace transform of Equation (11) we have

$$L^{-1}\left[\underline{y}(t,\alpha)\right]=\frac{1}{k}\underline{y}+\frac{1}{k}\underline{y}\left(\frac{s}{ak-s^2+bks}\right)+\left(\sinh t\sqrt{\frac{1}{4}b^2k^2+ak}\right)\frac{e^{\frac{1}{2}bkt}}{\sqrt{\frac{1}{4}b^2k^2+ak}} -\left(\frac{s}{ak-s^2+bks}\right)-\underline{y}\left(\sinh t\sqrt{\frac{1}{4}b^2k^2+ak}\right)\frac{e^{\frac{1}{2}bkt}}{\sqrt{\frac{1}{4}b^2k^2+ak}} +b\underline{y}\left(\sinh t\sqrt{\frac{1}{4}b^2k^2+ak}\right)\frac{e^{\frac{1}{2}bkt}}{\sqrt{\frac{1}{4}b^2k^2+ak}}+ak\underline{x}\left(\sinh t\sqrt{\frac{1}{4}b^2k^2+ak}\right)\frac{e^{\frac{1}{2}bkt}}{\sqrt{\frac{1}{4}b^2k^2+ak}} \quad (12)$$

Also, solving Equation (10) we have

$$L\left[\overline{y}(t,\alpha)\right]=\frac{\left[s^2+ks(a+bs)\right]\overline{x}_0(\alpha)+\left[s-(a+s)\right]\overline{y}_0(\alpha)}{s^3-ks(a+bs)} \quad (13)$$

Taking the inverse Laplace transform of Equation (13) we have

$$L^{-1}\left[\overline{y}(t,\alpha)\right]=\frac{1}{k}\overline{y}+\frac{1}{k}\overline{y}\left(\frac{s}{ak-s^2+bks}\right)+\left(\sinh t\sqrt{\frac{1}{4}b^2k^2+ak}\right)\frac{e^{\frac{1}{2}bkt}}{\sqrt{\frac{1}{4}b^2k^2+ak}} -\left(\frac{s}{ak-s^2+bks}\right)-\overline{y}\left(\sinh t\sqrt{\frac{1}{4}b^2k^2+ak}\right)\frac{e^{\frac{1}{2}bkt}}{\sqrt{\frac{1}{4}b^2k^2+ak}} +b\overline{y}\left(\sinh t\sqrt{\frac{1}{4}b^2k^2+ak}\right)\frac{e^{\frac{1}{2}bkt}}{\sqrt{\frac{1}{4}b^2k^2+ak}}+ak\overline{x}\left(\sinh t\sqrt{\frac{1}{4}b^2k^2+ak}\right)\frac{e^{\frac{1}{2}bkt}}{\sqrt{\frac{1}{4}b^2k^2+ak}} \quad (14)$$

Case 2. If $a \geq 0$ and $b < 0$

Putting case 2 into Equation (8) we have

$$s^3L\left[\overline{y}(t,\alpha)\right]-s^2\underline{x}_0(\alpha)-s\underline{y}_0(\alpha)-\underline{z}_0(\alpha)= ks^2\left[a+bs\right]L\left[\underline{y}(t,\alpha)\right]-ks(a+b)\underline{x}_0(\alpha)-(a+s)\underline{y}_0(\alpha)-\underline{z}_0(\alpha) \quad (15)$$

$$s^3L\left[\underline{y}(t,\alpha)\right]-s^2\overline{x}_0(\alpha)-s\overline{y}_0(\alpha)-\overline{z}_0(\alpha)=ks^2\left[a+bs\right]L\left[\overline{y}(t,\alpha)\right]-ks(a+b)\overline{x}_0(\alpha) \quad (16)$$

From Equation (15) we have

$$s^3L\left[\overline{y}(t,\alpha)\right]-ks^2\left[a+bs\right]L\left[\underline{y}(t,\alpha)\right]= \left[s^2-ks(a+b)\right]\underline{x}_0+\left[s-(a+s)\right]\underline{y}_0(\alpha) \quad (17)$$

This implies that

$$s^3L\left[\overline{y}(t,\alpha)\right]-ks^2\left[a+bs\right]L\left[\underline{y}(t,\alpha)\right]=A_1(t,\alpha) \quad (18)$$

Denoting that

$A_1(t,\alpha)=\left[s^2-ks(a+b)\right]\underline{x}_0+\left[s-(a+s)\right]\underline{y}_0(\alpha)$.

Similarly, from Equation (16) we have

$$s^3L\left[\underline{y}(t,\alpha)\right]-ks^2\left[a+bs\right]L\left[\overline{y}(t,\alpha)\right] =\left[s^2-ks(a+b)\right]\overline{x}_0+\left[s-(a+s)\right]\overline{y}_0(\alpha). \quad (19)$$

This implies that

$$s^3L\left[\underline{y}(t,\alpha)\right]-ks^2\left[a+bs\right]L\left[\overline{y}(t,\alpha)\right]=A_2(t,\alpha) \quad (20)$$

Denoting that

Now, solving Equations (18) and (20) we have (21)

$$L\left[\overline{y}(t,\alpha)\right]=\frac{s^3A_1(t,\alpha)+ks^2(a+bs)A_2(t,\alpha)}{s^6-k^2s^4(a+bs)} . \quad (21)$$

Taking the inverse Laplace transform of Equation we have

$$L\left[\overline{y}(t,\alpha)\right]=\frac{1}{k}A_2e^{\frac{1}{2}bk^2t}\frac{\sinh t\sqrt{\frac{1}{4}b^2k^4+ak^2}}{\sqrt{\frac{1}{4}b^2k^4+ak^2}}-\frac{1}{k}tA_2-\frac{1}{ak^2}A_1\left(\frac{1}{ak^2-s^2+bk^2s}\right)$$
$$-\frac{1}{ak^2}A_1-\frac{1}{a}bA_1e^{\frac{1}{2}bk^2t}\frac{\sinh t\sqrt{\frac{1}{4}b^2k^4+ak^2}}{\sqrt{\frac{1}{4}b^2k^4+ak^2}} \tag{22}$$

Similarly,

$$L\left[\underline{y}(t,\alpha)\right]=\frac{ks^2(a+bs)A_1(t,\alpha)+s^3A_2(t,\alpha)}{s^6-k^2s^4(a+bs)}\ . \tag{23}$$

Taking the inverse Laplace transform of Equation (23) we have

$$L\left[\underline{y}(t,\alpha)\right]=\frac{1}{k}A_1e^{\frac{1}{2}bk^2t}\frac{\sinh t\sqrt{\frac{1}{4}b^2k^4+ak^2}}{\sqrt{\frac{1}{4}b^2k^4+ak^2}}-\frac{1}{k}tA_1-\frac{1}{ak^2}A_2\left(\frac{1}{ak^2-s^2+bk^2s}\right)$$
$$-\frac{1}{ak^2}A_2-\frac{1}{a}bA_2e^{\frac{1}{2}bk^2t}\frac{\sinh t\sqrt{\frac{1}{4}b^2k^4+ak^2}}{\sqrt{\frac{1}{4}b^2k^4+ak^2}} \tag{24}$$

Case 3. If $a<0$ and $b\geq 0$

$$s^3L\left[\underline{y}(t,\alpha)\right]-s^2\underline{x}_0(\alpha)-s\underline{y}_0(\alpha)-\overline{z}_0(\alpha)=$$
$$ks^2\left[a+bs\right]L\left[\underline{y}(t,\alpha)\right]-ks(a+b)\underline{x}_0(\alpha)-(a+s)\underline{y}_0(\alpha)-\overline{z}_0(\alpha) \tag{25}$$

$$s^3L\left[\overline{y}(t,\alpha)\right]-s^2\overline{x}_0(\alpha)-s\overline{y}_0(\alpha)-\underline{z}_0(\alpha)=$$
$$ks^2\left[a+bs\right]L\left[\overline{y}(t,\alpha)\right]-ks(a+b)\overline{x}_0(\alpha)-(a+s)\overline{y}_0(\alpha)-\underline{z}_0(\alpha) \tag{26}$$

Now, solving Equations (25) and (26) we have

$$L\left[\underline{y}(t,\alpha)\right]=\frac{\left[s^2-ks(a+bs)\right]\underline{x}_0(\alpha)+\left[s-(a+s)\right]\underline{y}_0(\alpha)}{s^3-ks^2(a+bs)} \tag{27}$$

Taking the inverse Laplace transform of Equation we have

$$L^{-1}\left[\underline{y}(t,\alpha)\right]=\frac{1}{k}t\underline{y}_0+\frac{1}{ak^2}\left(\underline{y}_0-k+k\underline{y}_0+ak^2\underline{x}_0-bk\underline{y}_0\right)$$
$$+\frac{1}{ak^2}\frac{e^{ak\frac{t}{bk-1}}}{bk-1}\left(\underline{y}_0-k-ak^2+bk^2+k\underline{y}_0+ak^2\underline{x}_0-bk^2\underline{y}_0+b^2k^2\underline{y}_0-2bk\underline{y}_0\right) \tag{28}$$

Also, solving Equation (26) we have

$$L\left[\overline{y}(t,\alpha)\right]=\frac{\left[s^2-ks(a+bs)\right]\overline{x}_0(\alpha)+\left[s-(a+s)\right]\overline{y}_0(\alpha)}{s^3-ks^2(a+bs)}\ . \tag{29}$$

Taking the inverse Laplace transform of Equation (29) we have

$$L^{-1}\left[\overline{y}(t,\alpha)\right]=\frac{1}{k}t\overline{y}_0+\frac{1}{ak^2}\left(\overline{y}_0-k+k\overline{y}_0+ak^2\overline{x}_0-bk\overline{y}_0\right)$$
$$+\frac{1}{ak^2}\frac{e^{ak\frac{t}{bk-1}}}{bk-1}\left(\overline{y}_0-k-ak^2+bk^2+k\overline{y}_0+ak^2\overline{x}_0-bk^2\overline{y}_0+b^2k^2\overline{y}_0-2bk\overline{y}_0\right) \tag{30}$$

Case 4. If $a<0$ and $b<0$

Putting case 2 into Equation (8) we have

$$s^3L\left[\underline{y}(t,\alpha)\right]-s^2\overline{x}_0(\alpha)-s\overline{y}_0(\alpha)-\overline{z}_0(\alpha)=$$
$$ks^2\left[a+bs\right]L\left[\underline{y}(t,\alpha)\right]-ks(a+b)\underline{x}_0(\alpha)-(a+s)\underline{y}_0(\alpha)-\underline{z}_0(\alpha) \tag{31}$$

$$s^3L\left[\overline{y}(t,\alpha)\right]-s^2\underline{x}_0(\alpha)-s\underline{y}_0(\alpha)-\underline{z}_0(\alpha)=$$
$$ks^2\left[a+bs\right]L\left[\overline{y}(t,\alpha)\right]-ks(a+b)\overline{x}_0(\alpha)-(a+s)\overline{y}_0(\alpha)-\overline{z}_0(\alpha) \tag{32}$$

Now, solving Equations (31) and (32) we have

$$L\left[\underline{y}(t,\alpha)\right]=\frac{s^2\overline{x}_0(\alpha)-ks(a+b)\underline{x}_0(\alpha)+s\overline{y}_0(\alpha)-(a+s)\underline{y}_0(\alpha)+\overline{z}_0(\alpha)+\underline{z}_0(\alpha)}{s^3-ks^2(a+bs)}\ . \tag{33}$$

Taking the inverse Laplace transform of Equation (33) we have

$$L^{-1}\left[\underline{y}(t,\alpha)\right]=$$
$$\frac{1}{a^2k^2}\left(a\underline{y}_0-\overline{z}_0-\underline{z}_0+a^2k^2\overline{x}_0+\underline{z}_0kb+bk\overline{z}_0+abk^2\underline{x}_0-abk\overline{y}_0\right)-\frac{1}{ak}t(\underline{z}_0+\overline{z}_0-a\underline{y}_0)$$
$$-\frac{1}{a^2k^2}\frac{e^{-ak\frac{t}{bk-1}}}{bk-1}(\underline{z}_0-\overline{z}_0-a\underline{y}_0+\underline{z}_0b^2k^2+b^2k^2\overline{z}_0-2\underline{z}_0bk-2bk\overline{z}_0-abk^2\overline{x}_0+ab^2k^3\underline{x}_0$$
$$+a^2bk^3\underline{x}_0-ab^2k^2\underline{y}_0+2abk\overline{y}_0) \tag{34}$$

Also, solving Equation (32) we have

$$L\left[\underline{y}(t,\alpha)\right]=\frac{s^2\underline{x}_0(\alpha)-ks(a+b)\overline{x}_0(\alpha)+s\underline{y}_0(\alpha)-(a+s)\overline{y}_0(\alpha)+\underline{z}_0(\alpha)+\overline{z}_0(\alpha)}{s^3-ks^2(a+bs)}\ . \tag{35}$$

Taking the inverse Laplace transform of Equation (35) we have

$$L^{-1}\left[\overline{y}(t,\alpha)\right]=$$
$$\frac{1}{a^2k^2}\left(a\overline{y}_0-\overline{z}_0-\underline{z}_0+a^2k^2\underline{x}_0+\underline{z}_0kb+bk\overline{z}_0+abk^2\overline{x}_0-abk\underline{y}_0\right)-\frac{1}{ak}t(\underline{z}_0+\overline{z}_0-a\underline{y}_0)$$
$$-\frac{1}{a^2k^2}\frac{e^{-ak\frac{t}{bk-1}}}{bk-1}(\underline{z}_0-\overline{z}_0-a\overline{y}_0+\underline{z}_0b^2k^2+b^2k^2\overline{z}_0-2\underline{z}_0bk-2bk\overline{z}_0-abk^2\underline{x}_0+ab^2k^3\overline{x}_0$$
$$+a^2bk^3\underline{x}_0-ab^2k^2\overline{y}_0+2abk\underline{y}_0) \tag{36}$$

3.2. Example

Example 1: (according to Sankar and Tapan, 2014): The population of fish in a pond was modeled by Sankar and Tapan (2014) with t measured in years with the number of fish in the pond as N_0 = (4,900, 5,000, 5,050) with k = 0.04 and p=2,000 where p and k are constants. In this work, we used the above example in comparison with the results obtained in Equations (34) and (36). Therefore, substituting a = 4,900, b = 5,000, p =2,000, s = 5,050, and k = 0.04 in Equations (34) and (36) where a, b, and s are population of the fish in the pond, and p and k are constants; then we have

$$L^{-1}\left[\dot{y}(t,\alpha)\right]=3.25(1-\alpha)+\sin h(63.9)t\frac{e^{200t}}{63.9}-1.30-(1-\alpha)\sin h(63.9)t\frac{e^{200t}}{63.9}$$

$$+5,000(1-\alpha)\sin h(63.9)t\frac{e^{200t}}{63.9}+4,000(1-\alpha)\sin h(63.9)t\frac{e^{200t}}{63.9} \quad (37)$$

$$L^{-1}\left[\dot{y}(t,\alpha)\right]=3.25(\alpha-1)+\sin h(63.9)t\frac{e^{200t}}{63.9}-1.30-(\alpha-1)\sin h(63.9)t\frac{e^{200t}}{63.9}$$

$$+5,000(\alpha-1)\sin h(63.9)t\frac{e^{200t}}{63.9}+4,000(\alpha-1)\sin h(63.9)t\frac{e^{200t}}{63.9} \quad (38)$$

Simplifying Equations (37) and (38) we have the equations presented as $\underline{y}(t,\alpha)$ and $\overline{y}(t,\alpha)$, respectively.

$$\underline{y}(t,\alpha) = 3.25\,(1-\alpha) + 2.2 \times 10^{67} - 1.30 - 2.2 \times 10^{67}\,(1-\alpha)t + 1.1 \times 10^{70}\,(1-\alpha)t + 8.8 \times 10^{71}\,(1-\alpha)t$$

$$\overline{y}(t,\alpha) = 3.25\,(1-\alpha) + 2.2 \times 10^{67} - 1.30 - 2.2 \times 10^{67}\,(1-\alpha)t + 1.1 \times 10^{70}\,(1-\alpha)t + 8.8 \times 10^{71}\,(1-\alpha)t$$

The two equations above are coded using C++ together with the data from the example and are presented in the dialogue box below. However, based on that, Tables 1 and 2 are generated and used to show Figure 1.

From Table 1, we see that $\underline{y}(t,\alpha)$ is an increasing function, $\overline{y}(t,\alpha)$ is a decreasing function, and $\underline{y}(t,\alpha) < \overline{y}(t,\alpha)$. This means that based on the results obtained in this work, the solution of Sankar and Tapan's (2014) model for the particular value of t is a strong solution.

From Table 2, we see that $\underline{y}(t,\alpha)$ is an increasing function, $\overline{y}(t,\alpha)$ is a decreasing function, and $\underline{y}(t,\alpha) < \overline{y}(t,\alpha)$. This also means that based on the results obtained in this work, the solution of Sankar and Tapan's (2014) model for the particular value of is a strong solution.

Also, from the results presented in Table 1, for the linear homogeneous case, we observed that the gap between the lower bound variable $\underline{y}(t,\alpha)$ and the upper bound variable $\underline{y}(t,\alpha)$ is very small, which was estimated to be 2.5 when compared with the gap of 15,503.992 obtained in the work of Sankar and Tapan (2014), as presented in Table 2. This indicates that by using the method obtained in this study, the result converges faster than that of Sankar and Tapan (2014).

Table 1. Linear homogeneous case for t = 2.

α	$\underline{y}(t,\alpha)$	$\overline{y}(t,\alpha)$
0	103.2	115.8
0.1	103.9	115.2
0.2	104.5	114.6
0.3	105.1	114
0.4	105.8	113.3
0.5	106.4	112.7
0.6	107	112.1
0.7	107.7	111.4
0.8	108.3	110.8

Table 2. Sankar and Tapan's (2014) case.

α	$\underline{y}(t,\alpha)$	$\overline{y}(t,\alpha)$
0	5,444.6181	60,694.2105
0.1	11,039.2047	61,326.5986
0.2	16,086.4565	61,552.5764
0.3	21,012.4065	61,376.2941
0.4	25,803.6025	60,914.7645
0.5	30,447.1563	60,231.0977
0.6	34,930.7908	59,378.3611
0.7	39,242.8848	58,404.7747
0.8	43,726.5193	59,230.5113

3.3. Discussion of the results on linear homogeneous third-order FODEs by FLTM

The results for linear homogeneous FODEs were indicated as Equation (1) and are subject to the initial condition (2). The fuzzy linear function in Equation (8) was considered and four cases emerged. In case 1,

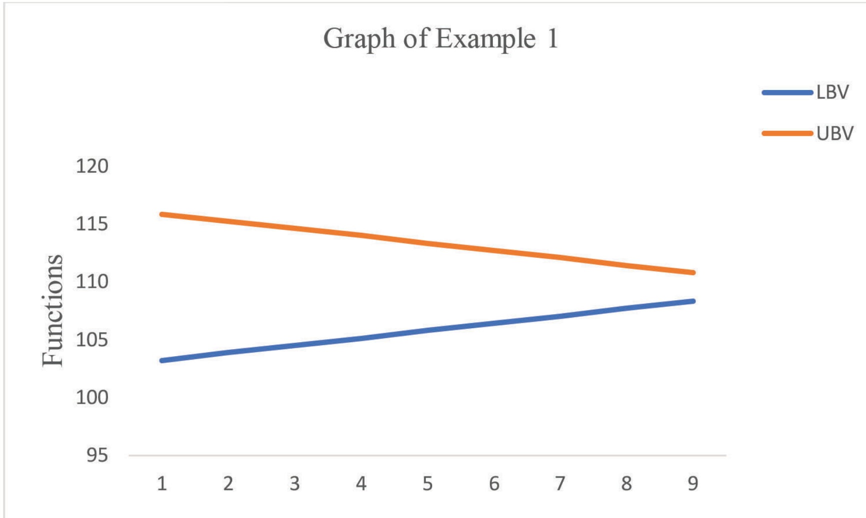

Figure 1. Graph of linear homogeneous FODEs.
In Figure 1, it is observed that the gap between the upper-bound variable $\overline{y}(t,\alpha)$ and lower-bound variable $\underline{y}(t,\alpha)$ decreases as α increases, which indicates a convergence.

$a \geq 0$ and $b \geq 0$ were considered and the conditions were applied to Equation (8). Consequently, Equations (12) and (14) were established. In case 2, $a \geq 0$ and $b < 0$ were considered and applied to Equation (8). The results obtained are indicated as Equations (22) and (24), respectively. In case 3, $a < 0$ and $b \geq 0$ were considered and applied to Equation (8), yielding the results in Equations (28) and (30), respectively. Finally, in case 4, $a < 0$ and $b < 0$ were considered and applied to Equation (8), resulting in Equations (34) and (36), respectively. However, the results above indicate that one can solve any physical problem related to linear homogeneous third-order FODEs using FLT.

3.4. On case of linear homogeneous FODEs for t = 2

C^{++} code was used on Equations (37) and (38), which yielded the numerical results in Table 1. The results showed that $\underline{y}(t,\alpha)$ is an increasing function, $\overline{y}(t,\alpha)$ is a decreasing function, and $\underline{y}(t,\alpha) < \overline{y}(t,\alpha)$. Therefore, the solution is a strong solution.

From the results presented in Table 1, in the linear homogeneous case we observed that the gap between the lower-bound variable $\underline{y}(t,\alpha)$ and the upper-bound variable $\overline{y}(t,\alpha)$ is very small, i.e., 2.5, when compared with the gap of 15,503.992 obtained in the work of Sankar and Tapan (2014), as presented in Table 2. This indicates that our result converges faster than that of Sankar and Tapan (2014).

3.5. On Figure 1 linear homogeneous FODEs

Figure 1 shows the graphical representation of the solutions of Tables 1 for values of α, lower-bound variable $\underline{y}(t,\alpha)$, and upper-bound variable $\overline{y}(t,\alpha)$. It is observed that the gap between the upper-bound variable $\overline{y}(t,\alpha)$ and the lower-bound variable $\underline{y}(t,\alpha)$ decreases as α increases, which indicates a convergence.

4. Conclusion

In this study, third-order linear FODEs were solved by FLTM. The results obtained were completely favorably when compared with the exact solution. The attractive properties of these methods are that they are implemented directly in a straightforward manner using the initial conditions at once without using the restrictive assumptions in the reviewed papers. Recommendations for further studies are made on the establishment of results for odd- and even-order FODEs.

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