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Laplace transform method for solving third-order linear nonhomogeneous ordinary differential equations in fuzzy environment

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ABSTRACT

In this paper, the solution procedure of a third-order linear nonhomogeneous ordinary differential equations in a fuzzy environment is described. Results of third-order linear nonhomogeneous are also established. An example is constructed using C^{++} codes on the elementary application of the population dynamics model to test the applicability of the set conditions. It is, therefore, recommended that an extension of the criteria to solve nonlinear third- or higher-order equations is considered as a motivation for further research.

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1. Introduction

Fuzzy differential equation (FDE) is a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models (Zhang et al., 2005). The studies of FDEs and fuzzy integral equations (FIEs) gained more attention from many authors in the past few years. For instance, Sankar and Tapan (2013) obtained some results for linear nonhomogeneous ordinary differential equations (ODEs) in a fuzzy environment.

The topics of FDEs and FIEs, which attracted growing interest for some time, particularly about fuzzy control, have been rapidly developed in recent years (Kaleva, 2005). Numerical algorithms for calculating approximate to these solutions were designed before discussing fuzzy differential and integral equations (Dubois and Prade, 1982). Recently, FDEs have also been used in many models such as Human Immunodeficiency Virus (HIV) model (Datta, 2003); decay model (Nachie, 2005); predator and prey model (Sankar and Tapan, 2015); and population models (Sankar and Tapan, 2013).

Fuzzy linear nonhomogeneous differential equation and initial value problem were extensively treated by Khastan et al. (2009), Lakshmikantham and Nieto (2003), Georgiou et al. (2005) and see also Goetschel and Voxman (1986).

This study aims at obtaining results for third-order linear nonhomogeneous ODEs using the fuzzy Laplace transform method (FLTM).

2. Methodology

This section presents the FLTM for solving third-order linear nonhomogeneous ODE by first presenting the concept of gH differentiability for third-order Fuzzy

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Ordinary Differential Equations (FODEs) and then the FLTM, respectively.

It is important to note that the concept of gH differentiability and FLTM for solving nonhomogeneous third-order FODEs are both the same as that of the homogeneous type. Therefore, we shall apply them in obtaining the results of this study.

2.1. Concept of gH-differentiability for third order FODEs

The third-order gH -derivative of a fuzzy valued function $f:[a,b]\rightarrow\mathfrak{R}_F$ at t_0 is defined as

$$f'''(t_0)=\lim_{h\rightarrow 0}\frac{f''(t_0+h)-_g Hf''(t_0)}{h}.$$

If $f'''(t_0)\in\mathfrak{R}_F$, we can say that $f''(t_0)$ is gH -differentiable at t_0.

1. $f''(t_0)$ is (i) gH -differentiable at t_0 if
   $f'''(t_0,\alpha)=\left(f_1''(t_0,\alpha),f_2''(t_0,\alpha)\right)$.
2. f is (i) gH -differentiable on (a,b) if
   $f'''(t_0,\alpha)=\left(f_2''(t_0,\alpha),f_1''(t_0,\alpha)\right)$.
3. f is (ii) gH -differentiable on (a,b), for all $\alpha\in[0,1]$.
4. $f''(t_0)$ is (ii) gH -differentiable at t_0 if
   $f'''(t_0,\alpha)=\left(f_2''(t_0,\alpha),f_1''(t_0,\alpha)\right)$.
5. f is (i) gH -differentiable on (a,b) then
   $f'''(t_0,\alpha)=\left(f_1''(t_0,\alpha),f_2''(t_0,\alpha)\right)$.
6. f is (ii) gH -differentiable on (a,b), for all $\alpha\in[0,1]$.

2.2. Laplace transform method for third-order FODEs

Suppose:

i. The derivative D takes a differentiable function f (defined on some interval (a,b)) and assigns to it a new function $\mathrm{D}f=f'$.

ii. The integral I takes a continuous function f (defined on some interval $[a,b]$) and assigns to it a new function

$$If(x)=\int_a^x f(t)\,dt.$$

iii. The multiplication operator M_φ, which multiplies any given function f on the interval $[a,b]$ by a fixed function φ on $[a,b]$, is a transform:

$$M_\varphi f(x)=\varphi(x)\cdot f(x).$$

The Laplace transform of the derivatives is

$$L[y']=\int_0^\infty e^{-px}y'(x)\,dx$$

and in general,

$$L\left[f^n(t)\right]=s^nf(s)-s^{n-1}f(0)-s^{n-2}f'(0)$$
$$-s^{n-3}f''(0)3s^{n-4}f'''(0)4\cdots-f^{n-1}(0).$$

3. Results and Discussion

3.1. Results

Consider the equation

$$y'''=ky+y_c \tag{1}$$

with initial conditions

$$y(0)=x_0$$
$$y'(0)=y_0$$
$$y''(0)=z_0 \tag{2}$$

$$L[y''']=kL[y]+L[y_c] \tag{3}$$

Take the Laplace transform of Equation (3) and simplify we have

$$s^3L\left[y(t,\alpha)\right]-s^2x_0(\alpha)-sy_0(\alpha)-z_0(\alpha)=\frac{k+1}{s^2}. \tag{4}$$

Now, we have the following alternatives for solving Equation (4).

Case 1. $a\geq 0$ and $b\geq 0$, Case 2. $a\geq 0$ and $b<0$, Case 3. $b\geq 0$ and $a<0$, Case 4. $b<0$ and . Suppose that the fuzzy linear function f is given by

$$ay''(t)+by'''(t) \tag{5}$$

Taking the Laplace transform of Equation (5) and simplify we have

$$f(t,\alpha)=s^2(a+bs)L\left[y(t,\alpha)\right]-s(a+s)x_0$$
$$(\alpha)-(a+s)y_0(\alpha)-z_0(\alpha)$$

Now, equating and simplifying Equations (5) and (6) we have

$$s^3L\left[y(t,\alpha)\right]-s^2x_0(\alpha)+sy_0(\alpha)+z_0(\alpha)-\frac{k+1}{s^2}=$$
$$s^2[a+bs]L\left[y(t,\alpha)\right]-s(a+b)x_0(\alpha)-(a+s)y_0(\alpha)-z_0(\alpha). \quad (7)$$

Now, applying the cases to Equation (7) we have

1. If $a \geq 0$ and $b \geq 0$
2. If $a \geq 0$ and $b < 0$
3. If $a < 0$ and $b \geq 0$
4. If $a < 0$ and $b < 0$

Case 1. $a \geq 0$ and $b \geq 0$

$$s^3L[\underline{y}(t,\alpha)]-s^2\underline{x}_0(\alpha)-s\underline{y}_0(\alpha)-\underline{z}_0(\alpha)-\frac{k+1}{s^2}=$$
$$s^5[a+bs]L[\underline{y}(t,\alpha)]-s^4(a+b)\underline{x}_0(\alpha)-s^3(a+s)\underline{y}_0(\alpha)-\underline{z}_0(\alpha) \cdot \quad (8)$$

$$s^3L[\bar{y}(t,\alpha)]-s^2\bar{x}_0(\alpha)-s\bar{y}_0(\alpha)-\bar{z}_0(\alpha)-\frac{k+1}{s^2}=$$
$$s^5[a+bs]L[\bar{y}(t,\alpha)]-s^4(a+b)\bar{x}_0(\alpha)-s^3(a+s)\bar{y}_0(\alpha)-\bar{z}_0(\alpha). \quad (9)$$

Solving Equation (8) we have

$$L[\underline{y}(t,\alpha)]=\frac{\frac{k+1}{s^2}[s^2-s^4(a+b)]\underline{x}_0(\alpha)+[s-s^3(a+s)]\underline{y}_0(\alpha)}{s^3-s^5(a+bs)} \quad (10)$$

Taking the inverse Laplace transform of Equation (10) we have

$$L^{-1}[\underline{y}(t,\alpha)]=$$
$$a+t-b\left(\frac{s}{as^2+bs^3-1}\right)+\underline{y}_0\left(\frac{s}{as^2+bs^3-1}\right)-a^2\left(\frac{s}{as^2+bs^3-1}\right)$$
$$-a^2\left(\frac{s}{as^2+bs^3-1}\right)-a\left(\frac{1}{as^2+bs^3-1}\right)$$
$$-b\left(\frac{1}{as^2+bs^3-1}\right)+\frac{1}{2}t^2(k+1)+ak+a\underline{x}_0\left(\frac{s}{as^2+bs^3-1}\right)-ab\left(\frac{s^2}{as^2+bs^3-1}\right)$$
$$-a^2k\left(\frac{s}{as^2+bs^3-1}\right)+b\underline{x}_0\left(\frac{s^2}{as^2+bs^3-1}\right)-bk\left(\frac{1}{as^2+bs^3-1}\right)$$
$$+a\underline{y}_0\left(\frac{1}{as^2+bs^3-1}\right)-abk\left(\frac{s^2}{as^2+bs^3-1}\right) \quad (11)$$

Also, solving Equation (9) we have

$$L[\bar{y}(t,\alpha)]=\frac{\frac{k+1}{s^2}[s^2-s^4(a+b)]\bar{x}_0(\alpha)+[s-s^3(a+s)]\bar{y}_0(\alpha)}{s^3-s^5(a+bs)}. \quad (12)$$

Taking the inverse Laplace transform of Equation (12) we have

$$L^{-1}[\bar{y}(t,\alpha)]=$$
$$a+t-b\left(\frac{s}{as^2+bs^3-1}\right)+\bar{y}_0\left(\frac{s}{as^2+bs^3-1}\right)-a^2\left(\frac{s}{as^2+bs^3-1}\right)$$
$$-a^2\left(\frac{s}{as^2+bs^3-1}\right)-a\left(\frac{1}{as^2+bs^3-1}\right)$$
$$-b\left(\frac{1}{as^2+bs^3-1}\right)+\frac{1}{2}t^2(k+1)+ak+a\bar{x}_0\left(\frac{s}{as^2+bs^3-1}\right)-ab\left(\frac{s^2}{as^2+bs^3-1}\right)$$
$$\left(\frac{s}{as^2+bs^3-1}\right)-ab\left(\frac{s^2}{as^2+bs^3-1}\right)$$
$$-a^2k\left(\frac{s}{as^2+bs^3-1}\right)+b\bar{x}_0\left(\frac{s^2}{as^2+bs^3-1}\right)-bk\left(\frac{1}{as^2+bs^3-1}\right)$$
$$-bk\left(\frac{1}{as^2+bs^3-1}\right)$$
$$+a\bar{y}_0\left(\frac{1}{as^2+bs^3-1}\right)-abk\left(\frac{s^2}{as^2+bs^3-1}\right) \quad (13)$$

Case 2. If $a \geq 0$ and $b < 0$. Putting Case 2 into Equation (7) we have

$$s^3L[\underline{y}(t,\alpha)]-s^2\underline{x}_0(\alpha)-s\underline{y}_0(\alpha)-\underline{z}_0(\alpha)-\frac{k+1}{s^2}=$$
$$s^5[a+bs]L[\bar{y}(t,\alpha)]-s^4(a+b)\underline{x}_0(\alpha)-s^3(a+s)\underline{y}_0(\alpha)-\underline{z}_0(\alpha)$$
$$s^5[a+bs]L[\bar{y}(t,\alpha)]-s^4(a+b)\underline{x}_0(\alpha)-s^3(a+s)\underline{y}_0(\alpha)-\underline{z}_0(\alpha) \quad (14)$$
$$s^5[a+bs]L[\bar{y}(t,\alpha)]-s^4(a+b)\underline{x}_0(\alpha)-s^3(a+s)\underline{y}_0(\alpha)-\underline{z}_0(\alpha) \quad (15)$$

From Equation (14) we have

$$s^3L[\bar{y}(t,\alpha)]-s^2\bar{x}_0(\alpha)-s\bar{y}_0(\alpha)-\bar{z}_0(\alpha)-\frac{k+1}{s^2}= \quad (16)$$

Denote

$$U_1(t,\alpha)=[s^2-s^4(a+b)]\underline{x}_0+[s-s^3(a+s)]\underline{y}_0(\alpha)+\frac{k+1}{s^2}.$$

Similarly, from Equation (15) we have

$$s^3L[\underline{y}(t,\alpha)]-s^5[a+bs]L[\bar{y}(t,\alpha)]=U_2(\alpha) \quad (17)$$

Denote

$$U_2(t,\alpha)=[s^2-s^4(a+b)]\bar{x}_0+[s-s^3(a+s)]\bar{y}_0(\alpha)+\frac{k+1}{s^2}.$$

Now, solving Equations (16) and (17) we have

$$L[\bar{y}(t,\alpha)] = \frac{s^3 U_1(t,\alpha)+s^5(a+bs)U_2(t,\alpha)}{s^6-s^{10}(a+bs)}. \quad (18)$$

Taking the inverse Laplace transform of Equation (18) we have

$$L^{-1}[\bar{y}(t,\alpha)] =$$

$$aU_2+\frac{1}{2}t^2U_1-a^2U_2\left(\frac{s^3}{as^4+bs^5-1}\right)-aU_1$$

$$\left(\frac{s}{as^4+bs^5-1}\right)-bU_1\left(\frac{s^2}{as^4+bs^5-1}\right) \quad (19)$$

$$L[\underline{y}(t,\alpha)] = \frac{s^5(a+bs)U_1(t,\alpha)+s^3U_2(t,\alpha)}{s^6-s^{10}(a+bs)} \quad (20)$$

Taking the inverse Laplace transform of Equation (20) we have

$$L^{-1}[\underline{y}(t,\alpha)] =$$

$$aU_1+\frac{1}{2}t^2U_2-a^2U_1\left(\frac{s^3}{as^4+bs^5-1}\right)$$

$$-aU_2\left(\frac{s}{as^4+bs^5-1}\right)-bU_2\left(\frac{s^2}{as^4+bs^5-1}\right)$$

$$-bU_1\left(\frac{1}{as^4+bs^5-1}\right)-abU_1\left(\frac{s^4}{as^4+bs^5-1}\right) \quad (21)$$

Case 3. If $a<0$ and $b\geq 0$. Putting Case 3 into Equation (7) we have

$$s^3L[\underline{y}(t,\alpha)] - s^2\underline{x}_0(\alpha) - s\underline{y}_0(\alpha) - \bar{z}_0(\alpha) - \frac{k+1}{s^2} =$$

$$s^5[a+bs]L[\underline{y}(t,\alpha)] - s^4(a+b)\underline{x}_0(\alpha) - s^3(a+s)\underline{y}_0(\alpha) - \bar{z}_0(\alpha) \quad (22)$$

$$s^3L[\bar{y}(t,\alpha)] - s^2\bar{x}_0(\alpha) - s\bar{y}_0(\alpha) - \underline{z}_0(\alpha) - \frac{k+1}{s^2} =$$

$$s^5[a+bs]L[\bar{y}(t,\alpha)] - s^4(a+b)\bar{x}_0(\alpha) - s^3(a+s)\bar{y}_0(\alpha) - \underline{z}_0(\alpha) \quad (23)$$

Now, solving Equation (22) we have

$$L[\underline{y}(t,\alpha)] = \frac{[s^2-s^4(a+bs)]\underline{x}_0(\alpha)+[s-s^3(a+s)]\underline{y}_0(\alpha)+\frac{k+1}{s^2}}{s^3-s^5(a+bs)}. \quad (24)$$

Taking the inverse Laplace transform of Equation (24) we have

$$L^{-1}[\underline{y}(t,\alpha)] =$$

$$\frac{1}{2}t^2(a+ak)-b\left(\frac{s}{as^2+bs^3-1}\right)-a\left(\frac{s}{as^2+bs^3-1}\right)+a^2k-a^3\left(\frac{s}{as^2+bs^3-1}\right)$$

$$-b\left(\frac{s^2}{as^2+bs^3-1}\right)-b^2\left(\frac{s}{as^2+bs^3-1}\right)-a\left(\frac{1}{as^2+bs^3-1}\right)+t(b+bk+1)$$

$$+\frac{1}{24}t^4(k+1)+a^2-a^2b\left(\frac{s^2}{as^2+bs^3-1}\right)+a\underline{x}_0\left(\frac{s}{as^2+bs^3-1}\right)$$

$$+b\underline{y}_0\left(\frac{s}{as^2+bs^3-1}\right)-a^3k\left(\frac{s}{as^2+bs^3-1}\right)-b^2k\left(\frac{s}{as^2+bs^3-1}\right)$$

$$+b\underline{y}_0\left(\frac{s}{as^2+bs^3-1}\right)-a^3k\left(\frac{s}{as^2+bs^3-1}\right)-b^2k\left(\frac{s}{as^2+bs^3-1}\right)$$

$$+b\underline{y}_0\left(\frac{s}{as^2+bs^3-1}\right)-a^3k\left(\frac{s}{as^2+bs^3-1}\right)-b^2k\left(\frac{s}{as^2+bs^3-1}\right)$$

$$-a^2bk\left(\frac{s^2}{as^2+bs^3-1}\right)-2abk\left(\frac{1}{as^2+bs^3-1}\right)+1 \quad (25)$$

Also, solving Equation (23) we have

$$L[\bar{y}(t,\alpha)] = \frac{[s^2-s^4(a+bs)]\bar{x}_0(\alpha)+[s-s^3(a+s)]\bar{y}_0(\alpha)+\frac{k+1}{s^2}}{s^3-s^5(a+bs)}. \quad (26)$$

Taking the inverse Laplace transform of Equation (26) we have

$$L^{-1}[\bar{y}(t,\alpha)] =$$

$$\frac{1}{2}t^2(a+ak)-b\left(\frac{s}{as^2+bs^3-1}\right)-a\left(\frac{s}{as^2+bs^3-1}\right)+a^2k-a^3\left(\frac{s}{as^2+bs^3-1}\right)$$

$$-b\left(\frac{s^2}{as^2+bs^3-1}\right)-b^2\left(\frac{s}{as^2+bs^3-1}\right)-a\left(\frac{1}{as^2+bs^3-1}\right)+t(b+bk+1)$$

$$+\frac{1}{24}t^4(k+1)+a^2-a^2b\left(\frac{s^2}{as^2+bs^3-1}\right)+a\bar{x}_0\left(\frac{s}{as^2+bs^3-1}\right)$$

$$+b\bar{x}_0\left(\frac{s^2}{as^2+bs^3-1}\right)-2ab\left(\frac{1}{as^2+bs^3-1}\right)+a\bar{y}_0\left(\frac{1}{as^2+bs^3-1}\right)$$

$$+b\bar{x}_0\left(\frac{s^2}{as^2+bs^3-1}\right)-2ab\left(\frac{1}{as^2+bs^3-1}\right)+a\bar{y}_0\left(\frac{1}{as^2+bs^3-1}\right)$$

$$-a^2bk\left(\frac{s^2}{as^2+bs^3-1}\right)-2abk\left(\frac{1}{as^2+bs^3-1}\right)+1 \quad (27)$$

Case 4. If $a<0$ and $b<0$. Putting Case 2 into Equation (7) we have

$$s^3L[\underline{y}(t,\alpha)] - s^2\underline{x}_0(\alpha) - s\bar{y}_0(\alpha) - \bar{z}_0(\alpha) - \frac{k+1}{s^2} =$$

$$s^5[a+bs]L[\underline{y}(t,\alpha)] - s^4(a+b)\bar{x}_0(\alpha) - s^3(a+s)\underline{y}_0(\alpha) - \underline{z}_0(\alpha) \quad (28)$$

$$s^3L[\bar{y}(t,\alpha)] - s^2\bar{x}_0(\alpha) - s\underline{y}_0(\alpha) - \underline{z}_0(\alpha) - \frac{k+1}{s^2} =$$
$$s^5[a+bs]L[\bar{y}(t,\alpha)] - s^4(a+b)\underline{x}_0(\alpha) - s^3(a+s)\bar{y}_0(\alpha) - \bar{z}_0(\alpha) \tag{29}$$

Now, solving Equation (28) we have

$$L[\underline{y}(t,\alpha)] = \frac{s^2\underline{x}_0(\alpha) - s^4(a+b)\bar{x}_0(\alpha) + s\bar{y}_0(\alpha) - (a+s)\underline{y}_0(\alpha) + \bar{z}_0(\alpha) + \underline{z}_0(\alpha) + \frac{k+1}{s^2}}{s^3 - s^5(a+bs)} \tag{30}$$

Taking the inverse Laplace transform of Equation (30) we have

$$L^{-1}[\underline{y}(t,\alpha)] =$$
$$a + \bar{z}_0 - \underline{x}_0\left(\frac{s}{as^2+bs^3-1}\right) - a^2\left(\frac{s}{as^2+bs^3-1}\right) - b\left(\frac{1}{as^2+bs^3-1}\right)$$
$$+\underline{y}_0\left(\frac{1}{as^2+bs^3-1}\right)$$
$$+\frac{1}{2}t^2(k+1) + ak - a\bar{z}_0\left(\frac{s}{as^2+bs^3-1}\right) - ab\left(\frac{s^2}{as^2+bs^3-1}\right)$$
$$-a^2k\left(\frac{s}{as^2+bs^3-1}\right) + a^2\bar{y}_0\left(\frac{s}{as^2+bs^3-1}\right) - b\underline{z}_0\left(\frac{s^2}{as^2+bs^3-1}\right)$$
$$-bk\left(\frac{1}{as^2+bs^3-1}\right) - abk\left(\frac{s^2}{as^2+bs^3-1}\right) + ab\underline{y}_0\left(\frac{s^2}{as^2+bs^3-1}\right) \tag{31}$$

Also, solving Equation (29)

$$L[\bar{y}(t,\alpha)] = \frac{s^2\bar{x}_0(\alpha) - s^4(a+b)\underline{x}_0(\alpha) + s\underline{y}_0(\alpha) - (a+s)\bar{y}_0(\alpha) + \underline{z}_0(\alpha) + \bar{z}_0(\alpha) + \frac{k+1}{s^2}}{s^3 - s^5(a+bs)} \tag{32}$$

Taking the inverse Laplace transform of Equation (32) we have

$$L^{-1}[\bar{y}(t,\alpha)] =$$
$$a^2\left(\frac{s}{as^2+bs^3-1}\right) - |\underline{z}_0 - a + b\left(\frac{1}{as^2+bs^3-1}\right) - \bar{x}_0\left(\frac{1}{as^2+bs^3-1}\right)$$
$$+\frac{1}{2}t^2(k+1) - ak - a\bar{y}_0 + a\bar{z}_0\left(\frac{s}{as^2+bs^3-1}\right) - \frac{1}{b}t\underline{x}_0(a+b)$$
$$+ab\left(\frac{s^2}{as^2+bs^3-1}\right) + a^2k\left(\frac{s}{as^2+bs^3-1}\right) + a\bar{x}_0\left(\frac{s}{as^2+bs^3-1}\right)$$
$$+b\underline{z}_0\left(\frac{s}{as^2+bs^3-1}\right) + bk\left(\frac{1}{as^2+bs^3-1}\right) + a\underline{x}_0\bar{y}_0\left(\frac{s}{as^2+bs^3-1}\right)$$
$$-\frac{a}{b}\bar{x}_0\left(\frac{1}{as^2+bs^3-1}\right) + \frac{a^2}{b}\underline{x}_0\left(\frac{s^2}{as^2+bs^3-1}\right) + abk\left(\frac{s^2}{as^2+bs^3-1}\right)$$
$$+b\bar{x}_0\underline{y}_0\left(\frac{s^2}{as^2+bs^3-1}\right) \tag{33}$$

3.1.1. Example

Example 1 (Due to Sankar and Tapan, 2013): Bacteria are being cultured for the production of medication. A differential equation is presented with Bacteria being harvested at a rate of N per hour, with p_0 bacteria in the culture where

p_0 = (7,800, 8,000, 8,150), k = 0.2, N = 1,000, k and N are constants. In this study, we use the information presented in Sankar and Tapan (2013) model described above with the results obtained in Equation (7).

Therefore, substituting

a = 78,000, b = 8,000, p =1,000, s = 8,150, and k = 0.2 in Equations (31) and (33) we have

$$L^{-1}[\underline{y}(t,\alpha)] =$$
$$t + 1.9\times10^{18}(1+\alpha) + 2.4t^2 + 1.5\times10^{23}(1.\alpha) + 1.2\times10^{26}(1.\alpha)$$
$$+1.8\times10^{19}(1.\alpha) - 1.1\times10^{31} \tag{34}$$

$$L^{-1}[\bar{y}(t,\alpha)] =$$
$$t + 1.9\times10^{18}(\alpha-1) + 2.4t^2 + 1.51)0^{23}(\alpha-1) + 1.2\times10^{26}(\alpha-1)$$
$$+1.81)0^{19}(\alpha-1) - 1.1\times10^{31} \tag{35}$$

Simplifying Equations (34) and (35) we have the equations presented as $\underline{y}(t,\alpha)$ and $\bar{y}(t,\alpha)$, respectively.

$\bar{y}(t,\alpha) = t + 1.9 \text{ X } 10^{18}(1-\alpha) + 2.4t^2 + 1.5 \text{ X } 10^{23}(1-\alpha) + 1.2 \text{ X } 10^{26}(1-\alpha) + 1.8 \text{ X } 10^{19}(1-\alpha) - 1.1 \text{ X } 10^{31}$

$\underline{y}(t,\alpha) = t + 1.9 \text{ X } 10^{18}(\alpha-1) + 2.4t^2 + 1.5 \text{ X } 10^{23}(\alpha-1) + 1.2 \text{ X } 10^{26}(\alpha-1) + 1.8 \text{ X } 10^{19}(\alpha-1) - 1.1 \text{ X } 10^{31}$

The two equations presented above are used in generating the codes using C++. The results presented in Tables 1 and 2 are obtained using those codes. Figure 1 is an outcome of the tables.

From Table 1, we see that $\underline{y}(t,\alpha)$ is an increasing function, $\bar{y}(t,\alpha)$ is a decreasing function and $\underline{y}(t,\alpha) < \bar{y}(t,\alpha)$. Therefore, the solution of the

Sankar and Tapan (2013) model for the particular value of is a strong solution.

From Table 2, we see that $\underline{y}(t,\alpha)$ is an increasing function, $\bar{y}(t,\alpha)$ is a decreasing function and $\underline{y}(t,\alpha) < \bar{y}(t,\alpha)$. Therefore, the solution is strong solution.

From the results presented in Table 1, the linear nonhomogeneous case, we observed that the gap between the lower bound variable $\bar{y}(t,\alpha)$ and the upper bound variable $\underline{y}(t,\alpha)$ is very small, i.e., 1.4 when compared with the gap of 25.978 obtained in the work of Sankar and Tapan (2013) presented as Table 2. This also indicated that our method converges faster than the method of Sankar and Tapan (2013).

In Figure 1, it is observed that the gap between the upper-bound variable and lower- bound variable decreases as increases, which also indicates convergence.

3.2. Discussion

3.2.1. On solution of linear nonhomogeneous third-order FODEs by FLTM

Fuzzy linear function in Equation (1) is considered together with the initial condition (2) using FLTM. Four cases emerged with Case 1 for $a \geq 0$ and $b \geq 0$ considered and applied to Equation (7). Results obtained are presented as Equations (11) and (13). Case 2 is when $a \geq 0$ and $b < 0$ applied to Equation (7) which yielded the results as Equations (19) and (21), respectively. Case 3 when $a < 0$ and $b \geq 0$ are applied to Equation (7). The results obtained are presented in Equations (25) and (27). Case 4 is when $a < 0$ and $b < 0$ applied to Equation (7) as well and the result is what is presented as Equations (31) and (33). This indicated that the established results are applicable to any physical problem related to linear nonhomogeneous FODEs using FLTM.

3.2.2. Solutions of third-order linear nonhomogeneous FODEs *t* = 2

C^{++} code is used on Equations (34) and (35) which yielded the numerical results in Table 1 shows that $\underline{y}(t,\alpha)$ is an increasing function, $\bar{y}(t,\alpha)$ is a decreasing

Table 1. Linear nonhomogeneous case for *t* = 2.

a	$\underline{y}(t,\alpha)$	$\bar{y}(t,\alpha)$
0	19.7	−1
0.1	18.7	0
0.2	17.7	1
0.3	16.6	2.1
0.4	15.6	3.1
0.5	14.6	4.2
0.6	13.5	5.2
0.7	12.5	6.2
0.8	11.4	7.3

Table 2. Sankar and Tapan (2013) model.

a	$\underline{y}(t,\alpha)$	$\bar{y}(t,\alpha)$
0	8,916.5228	12,027.9651
0.1	9,124.4346	11,797.2103
0.2	9,336.5274	11,570.1611
0.3	9,552.8820	11,346.7614
0.4	9,773.5808	11,126.9561
0.5	9,998.7076	10,910.6908
0.6	10,228.3479	10,697.9120
0.7	10,462.5889	10,488.5669
0.8	10,675.3677	10,701.3457

function and $\underline{y}(t,\alpha) < \bar{y}(t,\alpha)$. Therefore, the solution is a strong solution.

From the results presented in Table 1, the linear nonhomogeneous case, we observed that the gap between the lower bound variable and the upper bound variable is very small, i.e., 4.1 when compared with the gap of 25.978 obtained in the work of Sankar and Tapan (2013) presented as Table 2. This also indicated that our method converges faster than the method of Sankar and Tapan (2013).

3.2.3. Graph of linear nonhomogeneous FODEs

Figure 1 shows graphical representation of the solutions of Table 1 for values of α, lower bound variable $\underline{y}(t,\alpha)$, and upper bound variable $\bar{y}(t,\alpha)$. It is observed that the gap between the upper-bound variable $\bar{y}(t,\alpha)$

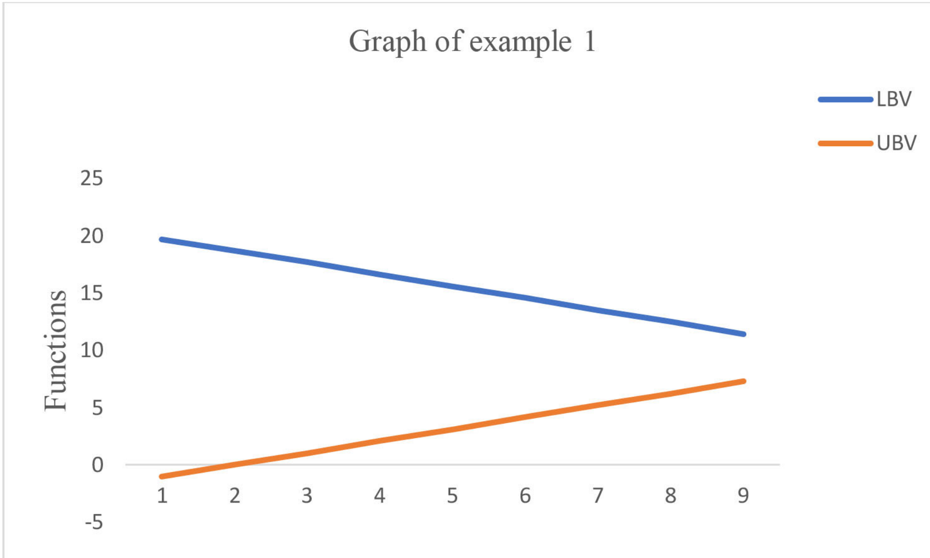

Figure 1. Graph of linear nonhomogeneous FODES.

and lower bound variable $\underline{y}(t,\alpha)$ decreases as α increases, which indicates convergence.

4. Conclusion

This work considers third-order linear nonhomogeneous ODEs in a fuzzy environment and FLTM is used to obtain the results. It is clearly seen that the results established in this work serve as a new contribution in related criteria for any form of linear equations. A related work concerning linear homogeneous case using FLTM can also be proposed. However, it is recommended that an extension of the criteria to solve nonlinear third- or higher-order equations is considered as a motivation for further research.

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