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Studying the Methods of Solutions for Some Nonlinear Stochastic Differential Equations

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Abstract

A variety of scientific fields, including physics, engineering, and finance, have recently used stochastic processes and stochastic calculus. A particular class of stochastic processes that are stochastically integral and frequently described as solutions to stochastic differential equations is the focus of stochastic calculus. Different methods such as: (integrate factor, heat equation, and Linearization) for some non-linear stochastic differential equations to find their analytic (exact) solution have been studied in this work. We also suggested a method to linearization the solution. The outcomes of applying the suggested method to the two models were identical to the outcomes of applying the integration factor method.

Keywords

Stochastic differential equation,
Stochastic integral,
Ito formula,
The integrated factor method,
Local linearization methods.

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1 - Introduction:

When the coefficients of a stochastic differential equation are linear and real functions, it is referred to as a linear stochastic differential equation (SDE); otherwise, it is called nonlinear.

The general form of any stochastic differential equation is an ordinary differential equation consisting of at least one term stochastic and therefore the solution is also a stochastic process(Mohammed & Khudair, 2023) .

A stochastic differential equation is essential for simulating physical systems in a variety of fields, including engineering, biology, and finance. Supposing if there exists the following ODE (Garcia, 2024; Salekin et al., 2026) .

$$\begin{cases} \dot{x}(t) = N\big(t, x(t)\big) \;\; ; \quad t > 0 \\ X(0) = x_0 \end{cases} \quad \dots (1)$$

Where x_0 is any initial (fixed) point $x_0 \in R^n$ and N(.) is any smooth function, Let equation (1) incorporate stochastic effects (Wiener process) to characterize the random phenomenon known as Brownian motion(Reis & Brites, 2025); such an equation takes the following form:

$$\dot{x}(t) = N\big(t, x(t)\big) + M(t, x(t))\delta(t) \;\; ; \quad t > 0 \quad \dots (2)$$

Here, $\delta(t)$ denotes the white noise process, which is the formal derivative of the Wiener process. Equation (2) can also be written as:

$$\frac{dx(t)}{dt} = N(t, x(t)) + M(t, x(t))\frac{dw(t)}{dt} \quad \dots (3)$$

Where $\frac{dw(t)}{dt} = \delta(t)$, $dw(t)$ denotes the derivative of the Wiener process. By multiplying both sides of equation (3) by dt , and we can get:

$$dx(t) = N\big(t, x(t)\big)dt + M(t, x(t))dw(t)$$
$$x(0) = x_0 \quad \dots (4)$$

Where N(.) and M(.) stand for the drift and diffusion coefficients, respectively. Eq. (4) is called stochastic differential equations(Salim & Abdulrhman, 2023). If $F\big(t, x(t)\big)$ is a smooth function for $0 \le t \le T$, we may obtain the following using the Taylor expansion rule:

$$dF\big(t, x(t)\big) = \left(\frac{\partial F}{\partial t} + N\big(t, x(t)\big)\frac{\partial F}{\partial x} + \frac{1}{2}M^2\big(t, x(t)\big)\frac{\partial^2 F}{\partial x^2}\right)dt + M\big(t, x(t)\big)\frac{\partial F}{\partial x}dw(t) \quad \dots (5)$$

eq. (5) is called the Ito formula where x(t) is the solution of eq. (4) (Bogoi et al., 2023; Saeed & Abdullah, 2025; Salim & Abdulrhman, 2023). M. Vas'kovskii investigated mixed-type stochastic differential equations with Hurst indices larger than 1/3 that were driven by standard and fractional Brownian movements in 2020(Vas'kovskii, 2020). Pourrahimi, F., E. Salavati, and S.A. MirHassani(Pourrahimi et al., 2025) studied Stochastic differential equations with G-Brownian motion and monotone coefficient.

In this work, the symbols of the aforementioned researchers, which are essential to the current study, were used. A new method was proposed to explain the need to find solutions to a nonlinear stochastic differential equation, comparing it with other methods, to study stochastic effects and the factors influencing them. We will apply the methods to stochastic differential equations of the logistic and square root kinds.

2- Preliminaries and Results:

In this section, some fundamental and necessary concepts for the non-linear SDE and the method of solution are given:

Definition.1 (The random variable)

A random variable is a variable with unknown numerical value that can take on, or represent, any possible element from a sample space. The elements of a sample space have probabilities associated (probability function). Thus, it is said that a random variable takes on various possible values with certain probability(Al-bagari, 2024).

Definition.2: (Stochastic process)

A stochastic process is a family of random variables $\{x(t), t \in \tau\}$ defined on a probability space (Ω, A, P) and indexed by a parameter t where t varies over a set τ. If the set τ is discrete, the stochastic process is called discrete. If the set τ is continuous, the stochastic process is called continuous(Al-bagari, 2024).

Definition.3 (Wiener Process)

Let $\{w(t)\}$ be a Wiener process in continuous time that satisfies the following requirements and is a (Brownian motion) across the interval $[0, T]$ (Saeed & Abdullah, 2025; Salim & Abdulrhman, 2023).

a. w (0) = 0
b. Supposing t, s ≥ 0, then w (t) - w (s) is distributed normally, with zero mean and variance$|t-s|$.
c. For $0 \leq s < t < u < v \leq T, w(t) - w(s)$
d. $w(u)$ is independent.

Definition.4 (Stochastic integral):

An integral that is defined as a sum greater than integration and that increases with time on the Wiener process trajectory is called a stochastic integral (McKean, 2024; Särkkä & Solin, 2019)That is:

$$\int_a^b q(t)dw(t) = \sum_{i=0}^{k-1} \delta i \left(w_{t_{i+1}} - w_{t_i}\right) \quad ...(6)$$

Where δi a real-valued stochastic process (sometimes, called a stochastic function or briefly a function) with respect to Wiener process $w(t)$ and $\{w(t)\}$is a Wiener process and $q(t)$ is a real-valued stochastic process, $q = \{q(t)\}_{a \leq t < b}$. The integral version of equation (4) may be expressed, as:

$$x(t) = x(0) + \int_0^t N(s, x(s))ds + \int_0^t M(s, x(s))dw(s) \quad ..(7)$$

The second integral is stochastic and requires careful attention to solve, while the first integral is deterministic. Also, the following requirements for $N(s, x(s))$ and $M(s, x(s))$ are needed before finding the analytic solution for the above equation.

E$(\int_0^t M^2(s, x(s))ds) < \infty$ then, for all $t \geq 0$ and $\int_0^t |N(s, x(s))|ds < \infty$

One of the most crucial characteristics of the stochastic integral is:

$\int_0^t w(s)\mathrm{d}w(s) = \frac{1}{2}\int_0^t d\left(w^2(s)\right) - \frac{1}{2}\int_0^t ds = \frac{1}{2}w^2(t) - \frac{t}{2}$ ...(8)

The general forms are:

$\int_0^t w^{\upsilon}(t)dw(t) = \frac{1}{\upsilon+1}w^{\upsilon+1}(t) + \sum_{j=0}^{\upsilon-2}(-1)^{j+1}\frac{\upsilon!}{2^{j+1}(\upsilon-j)!}w^{\upsilon-j}(t) + (-1)^{\upsilon}\frac{\upsilon!}{2^{\upsilon}}\frac{t}{2}$

(9)

Note: Supposing $\mathcal{F}_t = f(t)g(w_t)$ by using product role, there can be the following:

$$d\mathcal{F}_t = df(t)g(w_t) + f(t)dg(w_t)$$

$$= f'(t)g(w)dt + f(t)(g'(w_t)dw_t + \frac{1}{2}g''(w_t)dt)$$

Then

$$\int_a^b f(t)g'(w_t)dw_t = f(t)g(w_t) - \int_a^b f'(t)g(w)dt - \frac{1}{2}\int_a^b g''(w_t)dt$$

Definition.5 (Bernoulli equation)

The general form for the first order (Bernoulli equation (Oguadimma et al., 2026)) is as follows:

$$\frac{dx}{dy} + p(y)x = Q(y)x^n : n \neq 1$$

To reduce it to first order linear differential equation, the following method can be used (changing-variable) by assuming that:

$z = x^{1-n}$ Then we can get:

$$\frac{dz}{dy} = (1-n)x^{-n}\frac{dx}{dy}$$

So, the resulting transformation is a first-order linear differential equation of the form:

$$\frac{dz}{dy} + (1-n)p(y)z = (1-n)Q(y)$$

3- Main Result:

In this section, the exact solution of some non-linear stochastic differential equation will be presented and applied for nonlinear (SDE). Since, there is no general solution theory exists (See (Liu & Wang, 2021)), different methods for such process can be expressed.

3.1- Integrated factor:

Let us consider the initial value problem as articulated by eq (4), supposing that

$$Z(t) = exp(\int_0^t M(s)dw(s) - \frac{1}{2}\int_0^t M(s)^2\, ds)$$

The integrated factor is $\mathcal{F}(t) = Z^{-1}(t)$

Then $\mathcal{F}(t) = Exp(-\int_0^t M(s)dw(s) + \frac{1}{2}\int_0^t M(s)^2\, ds)$

So $\mathcal{F}(t)$ must satisfy:

$$d\mathcal{F}(t) = -M(t)\mathcal{F}(t)dw(t) + M^2(t)\mathcal{F}(t)dt$$

Let $Y(t) = Z(t)X(t)$

Then $Y(t) = Z^{-1}(t)X(t) = \mathcal{F}(t)X(t)$

Utilizing the Ito formula and the product role (Rezaeyan & Baloui, 2013) for $(\mathcal{F}(t)Y(t)) = d(X(t))$, we can get:

$$d\left(X(t)\right) = \mathcal{F}(t).N\left(t, Z(t)X(t)\right)dt$$

Where $\frac{dX(t)}{dt} = \mathcal{F}(t).N(t, Y(t));$ Since $Y(t) = Z(t)X(t)$

Then

$$X(t) = X(t_0) + \int_{t_0}^t \mathcal{F}(t)N\left(t, Z(t)X(t)\right)ds$$

3.2- Heat equation method:

The specific instance of the formula's inverse application is the heat equation method, which has the following definition:

Let $Q(t,y)$ be a solution of:

$$\frac{\partial Q(t,y)}{\partial t} + \frac{1}{2}\frac{\partial^2 Q(t,y)}{\partial y^2} = 0 \quad ...(10)$$

The heat equation without sources is represented by equation (10). The definition of the non-homogenous equation is:

$$\frac{\partial Q(t,y)}{\partial t} + \frac{1}{2}\frac{\partial^2 Q(t,y)}{\partial y^2} = G(t,y) \quad ...(11)$$

The heat equation with sources is represented by equation (11), where $G(t, y)$ is the density of heat sources and $Q(t, y)$ is the temperature at a location y. $G(t, y) = G(x)$ if a heat source is a time(Farkas & Deconinck, 2023).

Theorem.1 (Cauchy's theorem):

1): The solution $y(t)$ of the stochastic differential equation

$$dy(t) = h(t, y(t))dt + k(t, y(t))dw(t)$$

...(12)

is called exact if there is a differential function $\mathfrak{f}(t, y)$ such that:

$$\frac{\partial \mathfrak{f}(t,y)}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathfrak{f}(t,y)}{\partial y^2} = h(t, y) \ ; \quad \frac{\partial \mathfrak{f}(t,y)}{\partial y} = k(t, y)$$

Additionally, equation (12) may be expressed as follows if the exactness of the answer is satisfied:

$$dy(t) = \left(\partial_t \mathfrak{f}(t, y) + \frac{1}{2}\partial_y^2 \mathfrak{f}(t, y)\right) dt + \partial_y \mathfrak{f}(t, y)dw(t)$$

...(13)

2): Assuming that the following precise stochastic differential equation exists:

$$dy(t) = h(t, y(t))dt + k(t, y(t))dy(t)$$

then $h(t, x(t))$ and $k(t, x(t))$ must satisfy the following heat equation:

$$\frac{\partial h(t,y)}{\partial t} + \frac{1}{2}\frac{\partial^2 k(t,y)}{\partial y^2} = \frac{\partial h(t,y)}{\partial y}$$

...(14)

Note: If the condition in equation (14) does not satisfy then the stochastic differential equation is not exact so there is a need to use different methods to find out the solution to the given equation(Ogawa, 2004).

3.3- Linearization of stochastic differential equation:

3.3.1- Local Linearization Methods of Ozaki and Shoji

One example of a linearization approach is the local linearization techniques of Ozaki (Kirkby et al., 2025) and Shoji and Ozaki (Kirkby et al., 2025), where the linearization is selected in a particular manner. The following is a description of the fundamental concept of Ozaki's technique (Särkkä & Solin, 2019). Our goal is to approximate the following SDE's solution:

$$dy(t) = \mathfrak{f}(y)dt + dw(t)$$

...(15)

The goal is to use a linear SDE to estimate the solution on interval $[t,\ t + \Delta t]$, conditioned on $y(t)$

$$dy(t) = \mathcal{F}ydt + dw(t)$$

...(16)

Where the estimation of the linear equation is

$$\left.\begin{aligned} A(\Delta t) &= e^{\mathcal{F}\Delta t} \\ \Sigma(\Delta t) &= \int_0^{\Delta t} e^{(2\mathcal{F}(\tau-\Delta t)}q d\tau \\ &= \frac{q}{2F}[e^{2\mathcal{F}\Delta t} - 1] \end{aligned}\right\}$$

...(17)

giving the Gaussian approximation

$$p(y(t + \Delta t)|y(t)) \approx N(y(t + \Delta t)|A(\Delta t)y(t), \Sigma(\Delta t))$$

...(18)

let's now examine the noise-free equation to determine the approach.

$$\frac{d\xi}{dt} = \mathfrak{f}(\xi), \quad \xi(t) = y(t)$$

...(19)

Differentiating once gives

$$\frac{d^2\xi}{dt^2} = \mathfrak{f}'(\xi)\frac{d\xi}{dt}$$

...(20)

where the derivative of $\mathfrak{f}$ is denoted by $\mathfrak{f}'$. Now let's suppose that $\mathfrak{f}'(y(t))$ (x(t)) is a constant value for the derivative. After that, the differential equation (20) may be resolved to get

$$\frac{d\xi}{dt}(t + \Delta t) = \exp(\mathfrak{f}'(y(t))\Delta t)\frac{d\xi(t)}{dt} = \exp(\mathfrak{f}'(y(t))\Delta t)\,\mathfrak{f}(y(t))$$

...(21)

After that, integrating from 0 to Δt yields

$$\xi(t+\Delta t) = y(t) + \int_0^{\Delta t} \exp\big(f'(y(t))\tau\big) f(y(t)) d\tau$$
$$= y(t) + \frac{1}{f'(y(t))}\big[\exp\big(f'(y(t))\Delta t\big) - 1\big] f(y(t))$$
$$= \left(1 + \frac{f(y(t))}{y(t)f'(y(t))}\big[\exp\big(f'(y(t))\Delta t\big) - 1\big]\right) y(t) \quad ...(22)$$

Finally, we want to find F so that $\exp\big(f'(y(t))\Delta t\big)$ equals the coefficient of $y(t)$ in the previous, that is,

$$e^{(\mathcal{F}\Delta t)} = 1 + \frac{f(y(t))}{y(t)f'(y(t))}\big[\exp\big(f'(y(t))\Delta t\big) - 1\big] \quad ...(23)$$

which gives

$$\mathcal{F} = \frac{1}{\Delta t}\log(1 + \frac{f(y(t))}{y(t)f'(y(t))}\big[\exp\big(f'(y(t))\Delta t\big) - 1\big]) \quad ...(24)$$

Although it becomes more difficult and less practical, this approach can also be applied to multivariate SDEs. We obtain the following algorithm in the scalar case.

Algorithm.1:

Considering a scalar SDE of the type

$$dy(t) = f(y)dt + dw(t) \quad ...(25)$$

We use a linear time-invariant SDE to estimate its solution on interval $[t, t+\Delta t]$.

$$dy(t) = \mathcal{F}y(t)dt + dw(t) \quad ...(26)$$

were

$$\mathcal{F} = \frac{1}{\Delta t}\log(1 + \frac{f(y(t)}{x(t)f'(x(t))}\big[\exp\big(f'(x(t))\Delta t\big) - 1\big]) \quad ...(27)$$

Shoji and Ozaki improved Ozaki's local linearization approach to make it more applicable to multivariate applications (Särkkä & Solin, 2019). Our goal is to approximate a scalar SDE.

$$dy(t) = f(x,t)dt + dw(t) \quad ...(28)$$

with a linear SDE

$$dy(t) = Gy(t)dt + h(t)dt + kdt + dw(t) \quad ...(29)$$

Recall that the Ito formula for f gives

$$df(x,t) = \frac{\partial f(y,t)}{\partial t}dt + \frac{\partial f(y,t)}{\partial y}dx + \frac{1}{2}\frac{\partial^2 f(y,t)}{\partial y^2}dy^2$$
$$= \left[\frac{\partial f(y,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(y,t)}{\partial y^2}\right]dt + \frac{\partial f(y,t)}{\partial y}dy \quad ...(30)$$

Let us now assume that $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial^2 f}{\partial y^2}$ are constant. Then on the interval $[t,u]$, this gives

$$f(y(u),u) - f(y(t),t) = \left[\frac{\partial f(y,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(y,t)}{\partial y^2}\right](u-t) + \frac{\partial f(y,t)}{\partial t}(y(u) - y(t)) \quad ...(31)$$

We obtain the approximation

$$f(y(u),u) = G(t)y(u) + h(t)u + k(t) \quad ...(32)$$

when the terms are provided by

$$\left.\begin{aligned} G(t) &= \frac{\partial f(y,t)}{\partial y} \\ h(t) &= \frac{\partial f(y,t)}{\partial t} + \frac{q}{2}\frac{\partial^2 f(y,t)}{\partial y^2} \\ k(t) &= f(y(t),t) - \frac{\partial f(y,t)}{\partial y}y(t) - \left[\frac{\partial f(y,t)}{\partial t} + \frac{q}{2}\frac{\partial^2 f(y,t)}{\partial y^2}\right]t \end{aligned}\right\} \quad ...(33)$$

3.3.2- Suggested Method

When there is multiplicative noise (i.e., directly dependent on $y(t)$), it is difficult to identify an analytic integrative factor that reduces the SDE without adding further complexities caused by the Ito computation interactions. As a result, we present a method to linearize the solution. Let the following SDE be

$$dy(t) = f(y(t),t)dt + g(y(t),t)dw(t) \quad ...(34)$$

using the Ito formula

$$df = \left[f_t + \frac{1}{2}f_{yy}\right]dt + \left[f_y\right]dy$$
$$dg = \left[g_t + \frac{1}{2}g_{yy}\right]dt + \left[g_y\right]dy$$

Where f_t , g_t , f_y , g_y , g_{yy} , f_{yy} are constants.

Integrating equation (30) from t to u

$$f(y(t),u) = f(y(t),t) + \left[f_t + \frac{1}{2}f_{yy}\right](u-t) + [f_y](y(u) - y(t))$$

$\mathfrak{g}(y(t),u) = \mathfrak{g}(y(t),t) + \left[\mathfrak{g}_t + \frac{1}{2}\mathfrak{g}_{yy}\right](u-t) + [\mathfrak{g}_y](y(u) - y(t))$

then

$\mathfrak{f}(y(t),u) = G_1 y(u) + h_1(t)u + k_1(u)$

$\mathfrak{g}(y(t),u) = G_2 y(u) + h_2(t)u + k_2(u)$

so, linearization is

$d(y(t),u) = [G_1 y(u) + h_1(t)u + k_1(u)]dt + [G_2 y(u) + h_2(t)u + k_2(u)]dw(t)$ ... (35)

where $G_1 = \mathfrak{f}_y$, $G_2 = \mathfrak{g}_y$

$h_1(t) = \mathfrak{f}_t + \frac{1}{2}\mathfrak{f}_{yy}$

$h_2(t) = \mathfrak{g}_t + \frac{1}{2}\mathfrak{g}_{yy}$

$k_1(t) = \mathfrak{f}(y,t) - \mathfrak{f}_y y(t) - \left[\mathfrak{f}_t + \frac{1}{2}\mathfrak{f}_{yy}\right]t$

$k_2(t) = \mathfrak{g}(y,t) - \mathfrak{g}_y y(t) - [\mathfrak{g}_t + \frac{1}{2}\mathfrak{g}_{yy}]t$

So, the solution is

$y(t) = \emptyset_t(y(t) + \int_{t_0}^{t} \emptyset_s^{-1}(c(s) - b(s)ds)ds + \int_{t_0}^{t} \emptyset_s^{-1} d(s)\, dw(s)$ ...(36)

$c(s) = h_1(t) + k_1$

$b(s) = G_2 y$

$d(s) = h_2(t) + k_2$

So

$\emptyset_t = \exp\left(\int_{t_0}^{t}\left(G_1 y - \frac{G_1^2 y}{2}\right) ds + \int_{t_0}^{t} G_2 y\, dw(s)\right)$

4- Proposition:

The exact solution of the nonlinear SDE by the direct method (integrated factor) and by the suggested method have the same results.

This can be explained by taking some nonlinear stochastic differential equation models.

1. **The first model**

Let $y(t)$is to satisfy the solution of the following SDE:

$dy(t) = y(t)[a + by(t)]dt + py(t)dw(t)$

...(37) Where a ,b and p are constants

i. **The exactness:**

By applying Cauchy's theorem, we can have

$\mathfrak{f}(t,x) = ay + by^2$; $\mathfrak{g}(t,y) = py$

Since $\partial_y h \neq \partial_t k + \frac{1}{2}\partial_y^2 k$, hence the SDE is not exact. So, there will be a need for another method to solve it.

ii. **Integrating factor method**: Here

$$G(t) = e^{((\int_0^t c(s)dw(s) - \frac{1}{2}\int_0^t c(s)^2 ds)}$$

That is $G(t) = exp(pw_s - \frac{1}{2}p^2 t)$; $w(0) = 0$,

since $\mathcal{F}(t) = G^{-1}(t)$, then

$\mathcal{F}(t) = exp(-pw_s + \frac{1}{2}p^2 t)$

Supposing $y(t) = G(t)x(t)$

Then $x(t) = \mathcal{F}(t)y(t)$ and $N(t, y(t)) = N(t, G(t)x(t))$

From eq. (15), we have $N(t, X(t)) = ay + by^2$, then

$N(t, G(t)x(t)) = G(t)x(t)(a + bG(t)x(t)) = aG(t)x + bG^2(t)x^2(t)$

Since $d(\mathcal{F}(t)y(t)) = d(x(t))$, we can get:

$$d(x(t)) = \mathcal{F}(t).N(t, G(t)x(t))dt$$

$\frac{dx(t)}{dt} = \mathcal{F}(t)[aG(t)x + bG^2(t)x^2]$

Or

$\frac{dx(t)}{dt} = ax + bG(t)x^2$ ...(38)

Equation (38) is the form of a nonlinear Bernoulli equation, and by using the transformation ($z = x^{1-n}$, n=2), we can get:

$\frac{dz}{dt} + az = -b.exp(pw_s - \frac{1}{2}p^2 t)$ ...(39)

which is Bernoulli equation. The general solution of equation (39) is:

$z(t) = z(t_0) + e^{pw - at}\left[\frac{-2b\left(e^{at - \frac{1}{2}p^2 t} - 1\right)}{(p^2 - 2a)}\right]$

Then:

$$y(t) = [\frac{y_0\left(e^{\left(a-\frac{1}{2}p^2\right)t+pw_t}\right)}{1-bx_0[\frac{1}{\left(a-\frac{1}{2}p^2\right)}[\left(e^{\left(a-\frac{1}{2}p^2\right)t+pw_t}{}_{-1}\right)}] \quad ...(40)$$

iii. **The exact solution by suggested method:**

From equation (37) we assume

$dy(t) = y(t)[a + by(t)]dt + py(t)dw(t)$

$\mathfrak{f} = y(t)[a + by(t)]$, $\mathfrak{g} = py(t)$

$\Rightarrow \mathfrak{f}_t = \mathfrak{g}_t = 0$

$\mathfrak{f}_y = a + 2by(t)$ ، $\mathfrak{f}_{yy} = 2b$ ، $\mathfrak{g}_x = p$ ، $\mathfrak{g}_{yy} = 0$

$G_1 = \mathfrak{f}_y = a + 2by(t)$ ، $G_2 = \mathfrak{g}_y = p$

$\alpha_1(t) = \mathfrak{f}_t + \frac{1}{2}\mathfrak{f}_{yy} = b$

$\alpha_2(t) = \mathfrak{g}_t + \frac{1}{2}\mathfrak{g}_{yy} = 0$

$$b_1(t) = \mathfrak{f}(y(t), t) - \mathfrak{f}_y x(t) - \left[\mathfrak{f}_t + \frac{1}{2}\mathfrak{f}_{yy}\right]t = -by^2(t) - bt$$

$$b_2(t) = \mathfrak{g}(y(t), t) - \mathfrak{g}_y y(t) - [\mathfrak{g}_t + \frac{1}{2}\mathfrak{g}_{yy}]t = py(t) - py(t) = 0$$

as

$$d(y(u), u) = [G_1 y(u) + \alpha_1(t)u + b_1(u)]dt + [G_2 y(u) + \alpha_2(t)u + b_2(u)]dw(t)$$

$$dy(t) = [ay(t) + 2by(t) - by^2(t)]dt + [py(t)]dw(t)$$

$$y(t) = \emptyset_t(y(t) + \int_{t_0}^{t} \emptyset_s^{-1}(c(s) - b(s)ds)ds + \int_{t_0}^{t} \emptyset_s^{-1} d(s)\, dw(s)$$

$c(s) = \alpha_1(t) + b_1 \Rightarrow c(s) = b - by^2(t) - bt$

$b(s) = G_2 y(t) \Longrightarrow b(s) = py(t)$

$d(s) = \alpha_2(t) + b_2 = 0$

Then the exact solution is:

$$y(t) = [\frac{y_0\left(e^{\left(a-\frac{1}{2}p^2\right)t+pw_t}\right)}{1-bx_0[\frac{1}{\left(a-\frac{1}{2}p^2\right)}[\left(e^{\left(a-\frac{1}{2}p^2\right)t+pw_t}{}_{-1}\right)}] \quad ...(41)$$

Noting that the equation (41) is equivalence with equation (40) which explains that the same solution by different methods can be gotten.

2. **The second model**

Let y(t) is to satisfy the solution of the following SDE:

$$dy(t) = a(\mathrm{b} - y(t))dt + c\sqrt{y(t)}dw(t) \quad ...(42)$$

$$w_0 = 0$$

Where a , b and c are constants and W standard Brownian motion.

i. **The exactness:**

By applying Cauchy's theorem, we can have:

$\mathfrak{f}(t, y) = a(\mathrm{b} - y(t))$

$\mathfrak{g}(t, y) = b\sqrt{y(t)}$

Since $\partial_y h \neq \partial_t k + \frac{1}{2}\partial_y^2 k$, hence the SDE is not exact. So, there will be a need for another method to solve it.

ii. **The integrated factor method:**

$dy(t) = a(\mathrm{b} - y(t))dt + c\sqrt{y(t)}dw(t); x(0) = x_0$

Multiplying by e^{at}, we can get:

$$e^{at}dy(t) = e^{at}[a(b - y(t)]dt + ce^{at}\sqrt{y(t)}dw(t) = [abe^{at} - a\, e^{at}y(t)]dt + ce^{at}\sqrt{y(t)}dw(t)$$

Then $e^{at}dy(t) + ae^{at}y(t)dt = abe^{at}dt + ce^{at}\sqrt{y(t)}dw(t)$

Or equivalently

$$d\left(e^{at}y(t)\right) = abe^{at}dt + ce^{at}\sqrt{y(t)}dw(t)$$

by integrate from zero to t, we get

$$e^{at}y(t) - xy(0) = ab\int_0^t e^{at}ds + c\int_0^t e^{at}\sqrt{y(s)}dw(s)$$

Or, we can write it as:

$$y(t) = e^{-at}x(0) + \alpha ab\int_0^t ds + \gamma\int_0^t \sqrt{y(s)}dw(s)$$

The present study used the second integral, a stochastic integral (integrated by part), which yields:

$$\int_0^t [\sqrt{y(s)}dw(s) = \sqrt{y(t)}\, w(t) - \frac{1}{2}\int_0^t \frac{w(s)}{\sqrt{y(s)}}dx = \sqrt{y(t)}\, w(t) - \frac{1}{2}\sqrt{y(t)}\, w(t)$$

$$= \frac{1}{2}\sqrt{y(t)}\,w(t)$$

Then

$y(t) = e^{-a}y_0 + b(1 - e^{-at}) + \frac{c}{2}\sqrt{y(t)}\,w(t)$ …(43)

iii. **The exact solution by suggested method:**

From equation (42) we assume

$\mathfrak{f} = a(b - y(t))$, $\mathfrak{g} = c\sqrt{y(t)}$

$\Rightarrow \mathfrak{f}_t = \mathfrak{g}_t = 0$

$\mathfrak{f}_y = -\alpha$ ، $\mathfrak{f}_{yy} = 0$ ، $\mathfrak{g}_y = \frac{c}{2\sqrt{y(t)}}$ ، $\mathfrak{g}_{yy} = \frac{-c}{4\sqrt{y^3(t)}}$

$G_1 = \mathfrak{f}_y = -\alpha$ ، $G_2 = \mathfrak{g}_y = \frac{c}{2\sqrt{y(t)}}$

$\alpha_1(t) = \mathfrak{f}_t + \frac{1}{2}\mathfrak{f}_{yy} = 0$

$\alpha_2(t) = \mathfrak{g}_t + \frac{1}{2}\mathfrak{g}_{yy} = \frac{-c}{8\sqrt{y^3(t)}}$

$b_1(t) = \mathfrak{f}(y(t),t) - \mathfrak{f}_y y(t) - \left[\mathfrak{f}_t + \frac{1}{2}\mathfrak{f}_{yy}\right]t \Rightarrow$

$b_1(t) = \alpha\theta$

$b_2(t) = \mathfrak{g}(y(t),t) - \mathfrak{g}_y y(t) - [\mathfrak{g}_t + \frac{1}{2}\mathfrak{g}_{yy}]t \Rightarrow$

$b_2(t) = \frac{c}{2}\sqrt{y(t)} + \frac{ct}{8\sqrt{y^3(t)}}$

as

$$d(x(u),u) = [G_1x(u) + \alpha_1(t)u + b_1(u)]dt + [G_2x(u) + \alpha_2(t)u + b_2(u)]dw(t)$$

$dy(t) = [ab - ay(t)]dt + [c\sqrt{y(t)}]dw(t)$

$x(t) = \emptyset_t(x(t) + \int_{t_0}^{t} \emptyset_s^{-1}(c(s) - b(s)ds)ds + \int_{t_0}^{t} \emptyset_s^{-1} d(s)\,dw(s)$

$c(s) = \alpha_1(t) + b_1 \Rightarrow c(s) = ab$

$b(s) = G_2x(t) \Rightarrow b(s) = \frac{c}{2}\sqrt{y(t)}$

$d(s) = \alpha_2(t) + b_2$

$\quad = \frac{c}{2}\sqrt{y(t)}$

Then the exact solution is:

$y(t) = e^{-a}y_0 + b(1 - e^{-at}) + \frac{c}{2}\sqrt{y(t)}\,w(t)$ …(44)

Noting that the equation (43) is equivalence with equation (44) which explains that the same solution by different methods can be gotten.

5- Conclusion

The stochastic differential equation is called exact if it satisfies many conditions given by Cauchy's theorem. In the current study, after checking m for exactness it can be seen that the exact solution of the nonlinear SDE by the direct method (integrated factor) and by the suggested method have the same results. Such methods were applied to (the logistic model) and the square-root models, also discussed. The exactness of the solution could be very useful for needing another method. Therefore, when finding a solution to stochastic differential equations is difficult due to the presence of multiplier noise, we use the proposed method to linearization the solution.

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