https://doi.org/10.70882/f04kc345
SCIENCE FORUM (JOURNAL OF PURE AND APPLIED SCIENCES) 26, (4) (2026), 717 - 736

Science Forum (Journal of Pure and Applied Sciences)

Journal Homepage: https://atbuscienceforum.com.ng/index.php/jpas

On Deterministic Soft Sets: Algebraic Properties and Their Application to Electoral Decision Making

Uduak S. George[a], Edeghagba E. Elijah[a,b], Mustapha Ibrahim[a] Fatima Musa[c]
[a]Department of Mathematical Sciences, Sa'adu Zungur University, Bauchi, Nigeria
[b]Department of Computer Science, Al-Muhibbah Open University, Nigeria
[c]Department of Mathematics, Federal University of Health Science (FUHSA) Azare, Nigeria.

Abstract
Deterministic Soft Set (DSS) Theory extends classical soft set theory by incorporating deterministic external factors into parameterized representations of uncertainty. However, several fundamental structural properties required for a comprehensive mathematical framework remain undeveloped. This study strengthens the theoretical foundation of DSS Theory by formulating and establishing key algebraic and relational properties of deterministic soft sets. A reduction criterion is derived to characterize the conditions under which a deterministic soft set reduces to a classical soft set, thereby clarifying the relationship between the two frameworks. Formal definitions and associated properties are developed for deterministic equality and inclusion, null and absolute deterministic soft sets, and the deterministic NOT operation. The study further introduces deterministic soft equivalence relations and deterministic soft partitions and establishes their fundamental mathematical characteristics. It is demonstrated that deterministic soft inclusion satisfies reflexivity, antisymmetry, and transitivity and therefore defines a partial order on the collection of deterministic soft sets. Furthermore, a correspondence between deterministic soft equivalence relations and deterministic soft partitions is established, providing an important structural link within the DSS framework. The practical applicability of the extended theory is demonstrated using an electoral candidate-selection problem in which alternatives are evaluated with respect to specified parameters and deterministic external factors. A deterministic support-based decision model is applied to aggregate the resulting evaluations, identifying candidate c_2 as the preferred alternative under the prescribed conditions. The findings provide a more rigorous structural foundation for DSS Theory and broaden its applicability to parameter-dependent decision-making problems involving explicitly defined deterministic contextual factors.

Keywords
Deterministic Soft Set;
Soft Set Theory;
Deterministic Soft Equivalence Relation;
Deterministic Soft Partition;
Decision-Making.

EISSN: 3156-5190 (Online)
ISSN: 1119-4618 (Print)
JPAS
Faculty of Science, ATBU Bauchi

Corresponding Author: Edeghagba E. Elijah, Department of Mathematical Sciences, Sa'adu Zungur University, Bauchi, Nigeria
Email: edeghagbaelijah@sazu.edu.ng © 2026 Faculty of Science, ATBU Bauchi. All rights reserved

1 Introduction

Uncertainty is an inherent characteristic of numerous real-world systems and continues to present one of the most challenging problems in mathematical modelling, scientific computing, engineering, economics, medicine, artificial intelligence, environmental sciences, and decision sciences. Practical problems arising in these disciplines are frequently characterized by incomplete, inconsistent, vague, ambiguous, or imprecise information, making classical mathematical tools inadequate for representing real-life phenomena accurately. Classical set theory, pioneered by Cantor, provides a rigorous mathematical framework for representing precisely defined objects through binary membership, whereby an element either belongs or does not belong to a set. Although this framework has contributed significantly to the development of modern mathematics, its rigid binary structure limits its ability to represent the various forms of uncertainty encountered in practical applications. Consequently, the search for mathematical models capable of representing different manifestations of uncertainty has remained one of the most active research directions over the past six decades.

A major milestone in uncertainty modelling was achieved by Zadeh [20], who introduced the theory of fuzzy sets as a natural extension of classical set theory. Unlike classical sets, fuzzy sets assign each element a degree of membership within the interval [0, 1], thereby allowing gradual transitions between complete membership and complete non-membership. This simple yet revolutionary idea established the mathematical foundation of fuzzy logic and stimulated extensive research in approximate reasoning, computational intelligence, pattern recognition, image processing, optimization, expert systems, robotics, control engineering, and medical diagnosis [20, 21, 2, 4, 5, 6, 10, 7, 3]. Today, fuzzy set theory remains one of the most influential paradigms for modelling uncertainty and has inspired numerous theoretical developments and interdisciplinary applications.

The rapid growth of fuzzy mathematics led to several important extensions aimed at improving the expressive power of fuzzy models. Interval-valued fuzzy sets [9, 8, 13, 14, 15, 16], intuitionistic fuzzy sets [1], possibility theory [21], similarity measures [10, 19], entropy and distance measures [19], and approximate reasoning techniques [6, 17, 18] considerably enhanced the descriptive capability of fuzzy mathematics. These developments have produced powerful mathematical tools for handling uncertainty in diverse application areas, including intelligent decision support systems, autonomous control, industrial automation, financial modelling, computational biology, machine learning, and information retrieval. Nevertheless, most fuzzy methodologies rely fundamentally on the specification of appropriate membership functions, whose construction often depends on subjective expert knowledge or historical observations. In many practical situations, obtaining reliable membership values is difficult, thereby motivating the development of alternative uncertainty theories that avoid explicit membership assignments.

Another significant advancement in uncertainty modelling was achieved through the introduction of rough set theory by Pawlak [12]. Unlike fuzzy set theory, rough sets represent uncertainty using lower and upper approximations derived from indiscernibility relations, eliminating the need for membership functions or prior probabilistic information. Rough set theory has since become an important mathematical tool in knowledge discovery, feature selection, pattern recognition, data classification, machine learning, medical diagnosis, and information systems. Furthermore, numerous hybrid approaches combining fuzzy sets and rough sets have been proposed to exploit the complementary advantages of both methodologies in handling complex uncertainty. These developments demonstrate that uncertainty is inherently multidimensional and often requires different mathematical representations depending on the characteristics of the underlying problem.

Although fuzzy sets and rough sets have substantially advanced uncertainty modelling, many practical decision-making problems involve uncertainty that originates primarily from the parameters describing objects rather than from the objects themselves. In numerous real-life situations, the difficulty lies not in determining whether an object belongs to a particular class, but rather in identifying the appropriate parameters for describing that object. Traditional uncertainty models provide only limited flexibility in representing this form of parameter-dependent uncertainty because they focus principally on uncertainty associated with object membership or approximation. Consequently, a mathematical framework capable of explicitly incorporating parameterization into uncertainty modelling became increasingly necessary. This observation motivated Molodtsov [11] to introduce soft set theory, which represents uncertainty through parameterization instead of membership functions. In the soft set framework, every parameter is associated with an approximate subset of the universal set, thereby providing a simple and flexible representation of uncertain information without requiring probability distributions, membership grades, or auxiliary mathematical structures. The simplicity of this parameterized representation distinguishes soft set theory from many earlier uncertainty models and makes it particularly suitable for solving decision-making problems involving incomplete or qualitative information. Since its introduction, soft set theory has attracted remarkable attention from researchers worldwide and has evolved into one of the most active areas of uncertainty modelling, providing new perspectives for both theoretical investigations and practical applications.

The introduction of soft set theory opened a new direction in mathematical research by shifting the emphasis from uncertainty associated with objects to uncertainty associated with descriptive parameters. This paradigm has proven particularly effective in modelling multi-criteria decision-making problems, optimization, engineering design, medical diagnosis, information processing, economics, pattern recognition, and artificial intelligence. Consequently, soft set theory has become one of the fastest-growing branches of uncertainty modelling and continues to stimulate the development of increasingly sophisticated mathematical structures capable of representing complex decision environments.

The remarkable flexibility of soft set theory has stimulated extensive research directed toward strengthening both its theoretical foundations and its practical applicability. Shortly after the introduction of soft sets, Maji et al. [45, 43, 44] initiated systematic investigations into the algebraic properties of soft sets and demonstrated their effectiveness in solving decision-making problems under uncertainty. Their pioneering contributions established soft set theory as an important mathematical framework for representing parameter-dependent uncertainty and opened new research directions in information sciences and intelligent decision support systems. Subsequent studies focused on improving the computational efficiency and mathematical structure of soft sets. Chen et al. [32] introduced parameter reduction techniques that eliminate redundant parameters while preserving the consistency of decision outcomes. These techniques significantly improved the computational performance of soft set-based decision algorithms and have since become fundamental tools in soft computing. Later, Çağman and Enginoğlu [29, 30] developed soft matrix theory, providing an elegant matrix representation of soft sets that considerably simplified computational implementation and facilitated applications in optimization and multi-criteria decision making. Further developments involving generalized products, inverse soft matrices, cardinality soft matrices, and aggregation operators have continued to improve the computational efficiency and practical applicability of soft set models [27, 41, 46].

The success of classical soft set theory naturally inspired numerous extensions designed to improve its descriptive capability and applicability to increasingly complex uncertainty environments. Among the earliest of these developments were fuzzy soft sets [43], intuitionistic fuzzy soft sets [44], and interval-valued fuzzy soft sets [23]. These models combine the advantages of fuzzy representations with parameterized uncertainty, thereby providing more flexible mathematical structures for modelling complex decision environments. Other significant developments include possibility fuzzy soft sets [22], fuzzy parameterized soft sets [29, 30], soft expert sets [25], fuzzy soft expert sets [26], fuzzy soft multisets [24], and hypersoft sets [33]. Collectively, these extensions have substantially enriched the mathematical foundations of soft set theory and significantly expanded its scope of applications.

In parallel with these theoretical developments, soft set theory has found widespread application in decision sciences. Various researchers have proposed soft set-based algorithms for medical diagnosis, pattern recognition, engineering design, optimization, association rule mining, information retrieval, feature selection, supplier selection, financial decision making, and multi-criteria group decision analysis [38, 39, 35, 36, 40, 42]. These applications demonstrate the capability of soft sets to model parameterized uncertainty more naturally than many traditional uncertainty theories, particularly in situations where qualitative information plays a dominant role. More recently, increasing attention has been devoted to developing richer algebraic structures within the soft set framework. Research has investigated soft groups, soft rings, soft semigroups, soft ideals, soft topological spaces, and other algebraic systems, thereby demonstrating that soft set theory is not merely a decision-making tool but also an emerging branch of modern algebra with substantial theoretical potential. Furthermore, the introduction of full fuzzy parameterized soft sets by Elijah and Muhammad [34] provided a more comprehensive representation of fuzzy parameterized information and further expanded the theoretical landscape of soft computing. These developments illustrate the continuing evolution of soft set theory toward increasingly sophisticated mathematical structures capable of addressing more complex forms of uncertainty.

Despite these remarkable advances, existing soft set models primarily represent uncertainty through membership grades, parameter fuzzification, interval representations, possibility distributions, or expert opinions. However, many practical decision environments involve deterministic external conditions that directly influence the effectiveness of evaluation parameters without introducing additional fuzziness. Such deterministic influences remain largely unexplored within the existing soft set literature. This limitation motivated George et al. [37] to introduce

Deterministic Soft Set Theory (DSS), in which deterministic parameters are incorporated into the parameterization process through deterministic mappings. Their work established the basic framework of deterministic soft sets by defining deterministic approximation mappings together with the fundamental operations of complement, union, intersection, AND, and OR, and demonstrated the applicability of the proposed model to decision-making problems. Nevertheless, the algebraic development of deterministic soft sets remains at an early stage, leaving several fundamental structural concepts unexplored. The present study is motivated by this observation and seeks to strengthen the mathematical foundation of Deterministic Soft Set Theory through the introduction of new structural properties that naturally arise within the deterministic soft framework.

Furthermore, the applicability of the proposed theory is demonstrated through an electoral decision-making model that incorporates deterministic external factors affecting candidate selection. The remainder of this paper is organized as follows. Section 2 presents the preliminary concepts required for the subsequent developments. Section 3 introduces the new structural properties of deterministic soft sets together with their fundamental results. Section 4 illustrates the applicability of the proposed theory through an electoral decision-making model, while Section 5 concludes the paper and highlights possible directions for future research.

2 Preliminaries

This section briefly reviews the fundamental concepts and notation required throughout the paper. The definitions presented here are restricted to those directly relevant to the subsequent developments. Readers interested in the complete development of Deterministic Soft Set Theory are referred to George et al. [37].

Definition 2.1 (Molodtsov [11]) [Soft Set]. Let U be a universal set and let E be a set of parameters. A soft set over U is a pair (F, A), where $A \subseteq E$ and

$$F: A \to \mathcal{P}(U)$$

is a set-valued mapping assigning to each parameter a subset of the universe. The collection of all soft sets over U is denoted by $SS(U)$.

Definition 2.2 (George et al. [37]) [Deterministic Set]. A Deterministic Soft Set (DSS) over U is an ordered pair $F, M^{\#}A$, where

$$M^{\#}: A \to B$$

is a deterministic mapping and

$$F: M^{\#}A \to \mathcal{P}(U)$$

assigns to each deterministic parameter an approximate subset of U.

Throughout this paper, $D = F, M^{\#}A$ will denote a deterministic soft set unless otherwise stated.

Definition 2.3 (George et al. [37]) [Complement]. Let $D = F, M^{\#}A$ be a deterministic soft set. Its complement is defined by $D^c = (F^c, M^{\#}A)$, where

$$F^c\left(M^{\#}(a)\right) = U \backslash F\left(M^{\#}(a)\right)$$

for every $a \in A$.

Definition 2.4 (George et al. [37]) [Union]. Let $D_1 = F, M^{\#}A$ and $D_2 = G, M^{\#}B$ be deterministic soft sets. Their union is defined by

$$D_1 \cup D_2 = \left(H, P^{\#}(A \cup B)\right),$$

Where $H(\alpha) = F(\alpha) \cup G(\alpha)$ for every common deterministic parameter.

Definition 2.5 (George et al. [37]) [Intersection]. The intersection of $D_1 = F, M^{\#}A$ and $D_2 = G, M^{\#}B$ is

$$D_1 \cap D_2 = \left(H, P^{\#}(A \cap B)\right),$$

where $H(\alpha) = F(\alpha) \cap G(\alpha)$.

Definition 2.6 (George et al. [37]) [AND operation]. The AND operation between D_1 and D_2 is denoted by $D_1 \wedge D_2$, and is obtained by taking the Cartesian product of the corresponding deterministic parameter sets together with the intersection of their approximate values.

Definition 2.7 (George et al. [37]) [OR operation]. The OR operation between D_1 and D_2 is denoted by $D_1 \vee D_2$, and is obtained by taking the Cartesian product of the deterministic parameter sets together with the union of their approximate values.

The concepts and operations reviewed in this section provide the notation and mathematical framework required for the subsequent development of new structural properties of deterministic soft sets. In the next section, several fundamental concepts, including equality, null deterministic soft sets, absolute deterministic soft sets, deterministic soft equivalence relations, and deterministic soft partitions, are introduced and investigated.

3 Structural Properties of Deterministic Soft Sets

In this section we investigate the relationship between classical soft set and deterministic soft set and develop several fundamental structural properties of deterministic soft sets that were not considered in the original formulation of Deterministic Soft Set Theory.

3.1 Relationship Between Soft Sets and Deterministic Soft Sets

Let U be a universal set and E a set of parameters, let $A \subseteq E$, and let B be a set of deterministic factors. Let

$$\delta: B \to \{+, -\}$$

be a deterministic status function. The value $\delta(b) = +$ indicates that the deterministic factor b supports or activates the associated evaluation condition, whereas $\delta(b) = -$ indicates that it restricts, opposes, or deactivates that condition.

The deterministic parameter domain is denoted by $A_\delta^\#$. An element of this domain may be written as $a^{\delta(b)}b$, where $a \in A$ and $b \in B$. Thus, a deterministic soft set over U is represented by $D = F{:}A_\delta^\# B$, where

$$F{:}A_\delta^\# B \to \mathcal{P}(U).$$

The approximation is therefore indexed not only by the ordinary parameter a, but by the parameter together with the deterministic factor governing its interpretation.

Definition 3.1. Let $D = F{:}A_\delta^\# B$ be a deterministic soft set over U. Define the natural projection

$$\pi{:}A_\delta^\# B \to A$$

by

$$\pi\left(a_b^{\delta(b)}\right) = a.$$

The deterministic soft set D is called a deterministic enrichment of a soft set (G, A) if its approximations refine the parameterized information represented by G.

Theorem 1. Let $D = F{:}A_\delta^\# B$ be a deterministic soft set over U. Then D is formally a soft set over U with parameter set $A_\delta^\# B$.

Proof. Given the mapping $F{:}A_\delta^\# B \to \mathcal{P}(U)$ and (H, C) a soft set over U such that $H{:}C \to P(U)$. Taking $C = A_\delta^\# B$ and $H = F$, it follows that $(F, A_\delta^\# B)$ is a soft set. Hence every deterministic soft set is formally a soft set over its deterministic parameter domain.

Remark 1. The significance of Deterministic Soft Set Theory does not lie in abandoning the set-valued mapping structure of classical soft sets. Rather, it lies in enriching the parameter domain so that ordinary parameters are evaluated together with deterministic conditions.

Definition 3.2. Let $D = F: A_\delta^\# B$ be a deterministic soft set. Then D is said to reduce to a classical soft set over A if there exists a mapping

$$G: A \to \mathcal{P}(U)$$

such that

$$F(\gamma) = G(\pi(\gamma))$$

for every $\gamma \in A_\delta^\# B$.

Remark 2. The factorization condition $F = G \circ \pi$ means that the approximation depends only on the original parameter and not on the deterministic factor attached to it. In this case, the deterministic enrichment adds no new approximation information.

Theorem 2 (Reduction criterion). Let $D = F: A_\delta^\# B$ be a deterministic soft set, and $\pi: A_\delta^\# B \to A$ be the natural projection. Then D reduces to an ordinary soft set over A if and only if F is constant on every fibre of π. Equivalently,

$$\pi(\gamma_1) = \pi(\gamma_2) \Rightarrow F(\gamma_1) = F(\gamma_2)$$

for all $\gamma_1, \gamma_2 \in A_\delta^\# B$.

Proof. In one direction assume that D reduces to a soft set (G, A), then $F = G \circ \pi$. Suppose $\gamma_1, \gamma_2 \in A_\delta^\# B$ satisfy $\pi(\gamma_1) = \pi(\gamma_2)$, then

$$F(\gamma_1) = G(\pi(\gamma_1)) = G(\pi(\gamma_2)) = F(\gamma_2).$$

Thus, F is constant on every fibre of π.

Conversely, suppose that F is constant on every fibre of π. For each $a \in A$, choose an element $\gamma_a \in A_\delta^\# B$ such that $\pi(\gamma_a) = a$, and define

$$G(a) = F(\gamma_a).$$

Since F is constant on the fibre $\pi^{-1}(\{a\})$, this definition is independent of the chosen representative γ_a. Hence G is well defined.

Next let $\gamma \in A_\delta^\# B$. Since γ and $\gamma_{\pi(\gamma)}$ belong to the same fibre of π, it is the case that

$$F(\gamma) = F(\gamma_{\pi(\gamma)}) = G(\pi(\gamma)).$$

Thus, $F = G \circ \pi$. Hence D reduces to the classical soft set (G, A).

Corollary 1. Suppose that every parameter $a \in A$ is associated with exactly one neutral deterministic factor and that the parameter domain $A_\delta^\# B$ is naturally identified with A. Then every deterministic soft set $(F, A_\delta^\# B)$ reduces to the classical soft set (F, A).

Example 1. Let $U = \{c_1, c_2, c_3, c_4\}$ be a universe set of candidates, and let $A = \{competent, popular\}$. Let $B = \{constitutional\ eligibility, party\ clearance\}$. Consider the deterministic parameters

γ_1 = competent$^+$(constitutional eligibility),
γ_2 = competent$^+$(party clearance),
γ_3 = popular$^+$(constitutional eligibility),
γ_4 = popular$^+$(party clearance).

Define F by $F(\gamma_1) = F(\gamma_2) = \{c_1, c_2\}$ and $F(\gamma_3) = F(\gamma_4) = \{c_2, c_3, c_4\}$. The two deterministic factors attached to the parameter competent produce the same approximation, and the two deterministic factors attached to the parameter popular also produce the same approximation. Hence F is constant on every fibre of π. Therefore, the deterministic soft set reduces to the ordinary soft set (G, A) given by $G(competent) = \{c_1, c_2\}$ and $G(popular) = \{c_2, c_3, c_4\}$. Hence, the deterministic factors are present but do not contribute additional approximation information.

Example 2. Let $U = \{c_1, c_2, c_3, c_4\}$ be a universe set of candidates, and consider the parameter $a = popular$. Suppose that popularity is evaluated under the deterministic factors $b_1 = party\ zoning\ compliance$ and $b_2 = constitutional\ eligibility$. Let

$$\gamma_1 = \text{popular}^+(\text{party zoning compliance})$$

and

$$\gamma_2 = \text{popular}^+(\text{constitutional eligibility}).$$

Define $F(\gamma_1) = \{c_1, c_3\}$ and $F(\gamma_2) = \{c_1, c_2, c_4\}$. Since $\pi(\gamma_1) = \pi(\gamma_2) = popular$, the elements γ_1 and γ_2 belong to the same fibre of π. However,

$$F(\gamma_1) \neq F(\gamma_2).$$

Therefore, F is not constant on that fibre. By Theorem 2, there is no classical soft set over the original parameter set A that preserves both approximations.

This example explicitly exhibits the importance of the deterministic factor: the same ordinary attribute produces different approximation sets under different deterministic conditions.

Proposition 1. Let $D = F: A_\delta^\# B$ be a deterministic soft set. If there exist $\gamma_1, \gamma_2 \in A_\delta^\# B$ such that $\pi(\gamma_1) = \pi(\gamma_2)$ but $F(\gamma_1) \neq F(\gamma_2)$, then the deterministic component is essential, and D cannot be reduced to a classical soft set over A.

Proof. If D were reducible to a classical soft set over A, then by the reduction criterion, F would be constant on every fibre of π. However, γ_1 and γ_2 belong to the same fibre and satisfy $F(\gamma_1) \neq F(\gamma_2)$. This contradicts the required fibre constancy. Therefore, D cannot be reduced to a classical soft set over A.

3.2 Fundamental Structural Properties of Deterministic Soft Sets

In this subsection we develop some fundamental structural properties of deterministic soft sets, including equality, null and absolute structures, deterministic inclusion, the NOT operation, equivalence relations, and partitions. These concepts provide a unified framework for comparing, ordering, transforming, and classifying deterministic soft sets.

Two deterministic soft sets should not be regarded as equal merely because they produce the same subsets of the universe. Their parameter sets, deterministic factors, and deterministic status functions must also coincide.

Definition 3.3 (Equality). Let $D_1 = F: A_\delta^\# B$ and $D_2 = G: C_\varepsilon^\# K$ be two deterministic soft sets over the same universe U. Then $D_1 = D_2$ if and only if

$$A = C, \qquad B = K, \qquad \delta = \varepsilon,$$

and

$$F(\gamma) = G(\gamma)$$

for every $\gamma \in F: A_\delta^\# B$.

Equality of classical soft sets compares the parameter domain and the associated approximations. But deterministic soft equality additionally preserves the deterministic architecture attached to the parameters.

Definition 3.4 (Approximation equivalence). Let $D_1 = F: A_\delta^\# B$ and $D_2 = G: C_\varepsilon^\# K$ be two deterministic soft sets over U. Then D_1 and D_2 are said to be approximation-equivalent if

$$\{F(\gamma): \gamma \in A_\delta^\# B\} = \{G(\eta): \eta \in C_\varepsilon^\# K\}.$$

This is written as

$$D_1 \equiv_{\text{app}} D_2.$$

Remark 3. Approximation equivalence compares only the observable approximation sets, whereas deterministic equality also preserves the deterministic mechanism responsible for those approximations.

Theorem 3. Let D_1 and D_2 be deterministic soft sets over the same universe. If $D_1 = D_2$, then $D_1 \equiv_{\text{app}} D_2$.

Proof. Assume $D_1 = D_2$. It follows that their deterministic parameter systems coincide, and $F(\gamma) = G(\gamma)$ for every $\gamma \in F{:}A_\delta^\# B$. Consequently, since γ is arbitrary.

Then,

$$\{F(\gamma) : \gamma \in A_\delta^\# B\} = \{G(\gamma) : \gamma \in A_\delta^\# B\}.$$

Hence, $D_1 \equiv_{\text{app}} D_2$.

The converse of Theorem 3 does not necessarily hold as shown in Example 3 below.

Example 3. Let $U = \{c_1, c_2, c_3, c_4\}$ be a universe set of candidates, and let $a = eligible$ be an evaluation parameter. Consider the deterministic factors $b_1 = constitutional\ compliance$ and $b_2 = party$ screening approval. Define $D_1 = F{:}A_\delta^\# B$ and $D_2 = G{:}C_\varepsilon^\# K$, where

$F(a_{b_1}^+) = \{c_1, c_2, c_4\}$ and $G(a_{b_2}^+) = \{c_1, c_2, c_4\}$.

Both systems produce the same selected candidates. Therefore, $D_1 \equiv_{\text{app}} D_2$, because both deterministic soft sets generate the same approximation subset. But $D_1 \neq D_2$, given that $b_1 \neq b_2$.

The equality of the outcomes does not erase the distinction between the deterministic mechanisms that produced those outcomes. Constitutional compliance and party screening approval represent different decision constraints and may behave differently when the electoral conditions change.

Theorem 4. Equality of deterministic soft sets is an equivalence relation.

Proof. The proof follows easily from definition.

The concepts of null and absolute deterministic soft sets must preserve the deterministic parameter system while assigning either the empty set or the universal set to every deterministic parameter.

Definition 3.5 (Null Deterministic Soft Set). Let $D = (F{:}A_\delta^\# B)$ be a deterministic soft set over the universe U. The Null Deterministic Soft Set, denoted by $\mathbf{0}_D$, is defined by

$$\mathbf{0}_D = (F_\varnothing, A_\delta^\# B),$$

where

$$F_\varnothing(\gamma) = \varnothing$$

for every $\gamma \in A_\delta^\# B$.

Definition 3.6 (Absolute Deterministic Soft Set). Let $D = (F{:}A_\delta^\# B)$ be a deterministic soft set over the universe U. The Absolute Deterministic Soft Set, denoted by $\mathbf{1}_D$, is defined by

$$\mathbf{1}_D = (F_U, A_\delta^\# B),$$

where

$$F_U(\gamma) = U$$

for every $\gamma \in A_\delta^\# B$.

Remark 4. The null and absolute deterministic soft sets preserve exactly the same deterministic parameter system as the original deterministic soft set. Only the approximation mappings are replaced by the empty set and the universal set, respectively. Consequently, these structures represent extreme approximation states under an unchanged deterministic decision environment.

Example 4. Let U represent four electoral candidates. Suppose the evaluation parameters are competence and popularity, while the deterministic factors are constitutional eligibility and party clearance.

For the null deterministic soft set, $F_\emptyset(\gamma) = \emptyset$ for every deterministic parameter $\gamma \in A_\delta^\# B$. Hence, no candidate satisfies any evaluation criterion under any deterministic condition.

For the absolute deterministic soft set, $F_U(\gamma) = U$ for every deterministic parameter. Thus, every candidate satisfies every evaluation criterion under every deterministic condition.

Theorem 5. For every deterministic parameter system $A_\delta^\# B$, there exists exactly one null deterministic soft set and exactly one absolute deterministic soft set.

Within the deterministic soft framework, inclusion must preserve not only approximation mappings but also the deterministic parameter system.

Definition 3.7 (Deterministic Inclusion). Let $D_1 = (F: A_\delta^\# B)$ and $D_2 = (G: C_\varepsilon^\# K)$, be deterministic soft sets. Then $D_1 \preccurlyeq D_2$ if

$$A = C, \qquad B = K, \qquad \delta = \varepsilon,$$

and

$$F(\gamma) \subseteq G(\gamma)$$

for every $\gamma \in A_\delta^\# B$.

Theorem 6. Deterministic inclusion defines a partial order on the collection of all deterministic soft sets having a common deterministic parameter system.

Proof. Reflexivity easily holds since $F(\gamma) \subseteq F(\gamma)$ for every deterministic parameter. Next, antisymmetry holds because $D_1 \preccurlyeq D_2$ and $D_2 \preccurlyeq D_1$ imply $F(\gamma) = G(\gamma)$ for every deterministic parameter. Since the deterministic parameter systems are identical, $D_1 = D_2$. Lastly, transitivity follows immediately from ordinary subset inclusion. Hence, $(D, \preccurlyeq)$ forms a partially ordered set.

Proposition 2. If two deterministic soft sets are defined over different deterministic parameter systems, then they are incomparable under deterministic inclusion.

The complement of a deterministic soft set acts on the approximation subsets of the universe. In contrast, the NOT operation introduced in this section acts on the deterministic status attached to the parameters.

Let U be a universe, let A be a set of parameters, and let B be a set of deterministic factors. Suppose that

$$\delta: B \to \{+, -\}$$

is a deterministic status function. Define the opposite status function

$$\bar{\delta}: B \to \{+, -\}$$

by

$$\bar{\delta}(b) = \begin{cases} -, & \text{if } \delta(b) = +, \\ +, & \text{if } \delta(b) = -, \end{cases} \qquad b \in B.$$

Equivalently, $\bar{\bar{\delta}} = \delta$. For every deterministic parameter $\gamma = a_b^{\bar{\delta}(b)} \in A_\delta^\# B.$, define

$$\nu_\delta(\gamma) = a_b^{\bar{\delta}(b)} \in A_\delta^\# B.$$

Thus,

$$\nu_\delta: A_\delta^\# B \to A_\delta^\# B$$

is the canonical status-reversal mapping.

Definition 3.8 (NOT Deterministic Soft Set). Let $D = (F: A_\delta^\# B)$ be a deterministic soft set over U, with $F: A_\delta^\# B \to \mathcal{P}(U)$. The NOT deterministic soft set of D, denoted by $\neg D$, is defined by

$$\neg D = \left(F^\neg, A_\delta^\# B\right),$$

where $F^\neg: A_\delta^\# B \to \mathcal{P}(U)$ is given by

$$F^\neg\left(a_b^{\bar{\delta}(b)}\right) = F\left(a_b^{\delta(b)}\right)$$

for every $a \in A$ and $b \in B$.

Clearly, the NOT operation therefore reverses the deterministic status associated with every parameter without changing its underlying approximation subset. Hence, complement and NOT have fundamentally different interpretations: complement changes the approximate objects, whereas NOT changes the deterministic condition under which the approximation is interpreted.

Remark 5. If $D = (F: A_\delta^\# B)$, then the complement D^c and the NOT deterministic soft set $\neg D$ should not, in general, be identical. The complement operates on the approximation mapping according to

$$F^c(\gamma) = U \backslash F(\gamma),$$

whereas the NOT operation reverses the deterministic status

$$\delta(b) \mapsto \bar{\delta}(b).$$

Thus, the two operations capture two distinct types of negation in Deterministic Soft Set Theory.

Theorem 7. Let $D = (F: A^{\#}_{\delta}B)$ be a deterministic soft set over U. Then

$$\neg(\neg D) = D.$$

Proof. Let $D = (F: A^{\#}_{\delta}B)$ be a deterministic soft set over U. By the definition of NOT deterministic soft set, $\neg D = (F^{\neg}, A^{\#}_{\bar{\delta}}B)$, then

$$F^{\neg}\left(a_b^{\bar{\delta}(b)}\right) = F\big(a_b^{\delta(b)}\big).$$

Next, by applying the NOT operation on the NOT deterministic soft set, it gives:

$$\neg(\neg D) = \big((F^{\neg})^{\neg}, A^{\#}_{\bar{\bar{\delta}}}B\big).$$

It follows that $A^{\#}_{\bar{\bar{\delta}}}B = A^{\#}_{\delta}B$ since $\bar{\bar{\delta}} = \delta$. Furthermore, for every $a \in A$ and $b \in B$, $(F^{\neg})^{\neg}\big(a_b^{\delta(b)}\big) = F^{\neg}\left(a_b^{\bar{\delta}(b)}\right) = F\big(a_b^{\delta(b)}\big)$. Thus $(F^{\neg})^{\neg} = F$. Hence, $\neg(\neg D) = (F, A^{\#}_{\delta}B) = D$

Proposition 3. Let $D_1 = (F: A^{\#}_{\delta}B)$ and $D_2 = (G: C^{\#}_{\varepsilon}K)$ be deterministic soft sets over a common deterministic parameter system. If $D_1 \preccurlyeq D_2$, then

$$\neg D_1 \preccurlyeq \neg D_2.$$

Proof. Assume $D_1 \preccurlyeq D_2$. It follows that

$$F\big(a_b^{\delta(b)}\big) \subseteq G\big(a_b^{\delta(b)}\big)$$

for every $a \in A$ and $b \in B$. By definition,

$$F^{\neg}\left(a_b^{\bar{\delta}(b)}\right) = F\big(a_b^{\delta(b)}\big)$$

and

$$G^{\neg}\left(a_b^{\bar{\delta}(b)}\right) = G\big(a_b^{\delta(b)}\big).$$

Therefore,

$$F^{\neg}\left(a_b^{\bar{\delta}(b)}\right) \subseteq G^{\neg}\left(a_b^{\bar{\delta}(b)}\right),$$

hence $\neg D_1 \preccurlyeq \neg D_2$.

Remark 6. NOT is order-preserving, whereas ordinary set complement is order-reversing.

Theorem 8. Let $D_1 = (F, A^{\#}_{\delta}B)$ be a deterministic soft set over U. Then

$$(\neg D)^c = \neg(D^c).$$

Proof. Assume $\gamma = a_b^{\bar{\delta}(b)} \in A^{\#}_{\delta}B$, and put $\eta = a_b^{\bar{\delta}(b)} \in A^{\#}_{\delta}B$. By definition, the NOT operation changes the deterministic status from $\delta(b)$ to $\bar{\delta}(b)$, while the complement replaces the corresponding approximation set by its complement in U. For $\eta = a_b^{\bar{\delta}(b)}$,

Then,

$$((F^{\neg})^c)(\eta) = U \backslash F^{\neg}(\eta) = U \backslash F(\gamma).$$

In the reverse direction,

$$(F^c)^{\neg}(\eta) = F^c(\gamma) = U \backslash F(\gamma).$$

Therefore, $(F^{\neg})^c = (F^c)^{\neg}$. Both deterministic soft sets are defined over $A^{\#}_{\delta}B$. Therefore $(\neg D)^c = \neg(D^c)$.

Now we introduce deterministic soft relations in a manner that reflects the parameterized structure of a deterministic soft set.

Let $D = (F, A^{\#}_{\delta}B)$ be a deterministic soft set over a nonempty universe U, where $\mathrm{F}: A^{\#}_{\delta}B \to \mathcal{P}(U)$. For convenience, we use:

$$\Gamma = A^{\#}_{\delta}B.$$

Thus, for every $\gamma \in \Gamma$, the set

$$F(\gamma) \subseteq U$$

is the approximation associated with the enriched deterministic parameter γ.

Definition 3.9 (Deterministic Soft Relation). Let $D = (F, A^{\#}_{\delta}B)$ be a deterministic soft set over a nonmpty universe U. Where $\mathrm{F}: A^{\#}_{\delta}B \to \mathcal{P}(U)$. A deterministic soft relation on D is a mapping:

$$\mathcal{R}: \Gamma \to \bigcup_{\gamma \in \Gamma} \mathcal{P}(F(\gamma) \times F(\gamma))$$

Such that.

$$\mathcal{R}(\gamma) \subseteq F(\gamma) \times F(\gamma)$$

For every $\gamma \in \Gamma$.

Equivalently, we denote a deterministic soft relation as the indexed family:

$$\mathcal{R} = \{R_\gamma\}_{\gamma \in \Gamma},$$

where

$$\mathcal{R}_\gamma \subseteq F(\gamma) \times F(\gamma)$$

for every $\gamma \in \Gamma$.

Remark 7. For $x, y \in F(\gamma)$, $xR_\gamma y$ implies $(x, y) \in R\gamma$. Since the relation $R\gamma$ is indexed by the enriched deterministic parameter γ, the same pair of objects may be related under one deterministic condition and unrelated under another. This parameter dependence distinguishes deterministic soft relations from ordinary binary relations on U.

Definition 3.10 (Deterministic Soft Equivalence Relation). Let $D = (F, \Gamma)$ be a deterministic soft set and $\mathcal{R} = \{R_\gamma\}_{\gamma \in \Gamma}$, be a deterministic soft relation on D. Then $\mathcal{R}$ is said to be a deterministic soft equivalence relation on D if, for every $\gamma \in \Gamma$, the relation $R_\gamma \subseteq F(\gamma) \times F(\gamma)$ satisfies:

1. Reflexivity: $(x, x) \in R_\gamma$ for every $x \in F(\gamma)$.
2. Symmetry: $(x, y) \in R_\gamma \Rightarrow (y, x) \in R_\gamma$ for all $x, y \in F(\gamma)$.
3. Transitivity: $(x, y) \in R_\gamma \; and \; (y, z) \in R_\gamma \Rightarrow (x, z) \in R_\gamma$ for all $x, y, z \in F(\gamma)$.

Theorem 9. Let $D = (F, \Gamma)$ be a deterministic soft set. Then $\mathcal{R}$ is a deterministic soft equivalence relation on D if and only if R_γ is an ordinary equivalence relation on $F(\gamma)$ for every $\gamma \in \Gamma$.

Definition 3.11 (Deterministic Soft Equivalence Class). Let $D = (F, \Gamma)$ be a deterministic soft set and $\mathcal{R} = \{R_\gamma\}_{\gamma \in \Gamma}$, be a deterministic soft equivalence relation on D. For $\gamma \in \Gamma$ and $x \in F(\gamma)$, the deterministic soft equivalence class of x at γ is given as:

$$[x]R_\gamma = \{y \in F(\gamma): (x, y) \in R_\gamma\} = \{y \in F(\gamma): x \, R_\gamma \, y\}.$$

Theorem 10. Let $D = (F, \Gamma)$ be a deterministic soft set and $\mathcal{R} = \{R_\gamma\}_{\gamma \in \Gamma}$, be a deterministic soft equivalence relation on D. Then for any fixed $\gamma \in \Gamma$ and any $x, y \in F(\gamma)$, the following statements are equivalent:

1. $x \, R_\gamma \, y$;
2. $[x]R_\gamma = [y]R_\gamma$;
3. $[x]R_\gamma \cap [y]R_\gamma \neq \emptyset$.

Proof. Let $\gamma \in \Gamma$ be fixed.

(i) ⇒ (ii). Suppose $x \, R_\gamma \, y$ and choose $z \in [x]R_\gamma$. Then it is the case $x \, R_\gamma \, z$. By the symmetry property of $\mathcal{R}$, it follows that $x \, R_\gamma \, y$, and by transitivity, $x \, R_\gamma \, z$. Therefore, $z \in [y]R_\gamma$. Hence $[x]R_\gamma \subseteq [y]R_\gamma$. For the reverse inclusion we interchange x and y then $[x]R_\gamma \subseteq [y]R_\gamma$. Thus $[x]R_\gamma = [y]R_\gamma$

(ii) ⇒ (iii). Assume $[x]R_\gamma = [y]R_\gamma$. Then it follows easily that $[x]R_\gamma \cap [y]R_\gamma \neq \emptyset$

(iii) ⇒ (i). Suppose $[x]R_\gamma \cap [y]R_\gamma \neq \emptyset$. Then there exists $z \in F(\gamma)$ such that $z \in [x]R_\gamma$ and $z \in [y]R_\gamma$. Then it follows that, $x \, R_\gamma \, z$ and $y \, R_\gamma \, z$. Therefore, by the symmetry property, $z \, R_\gamma \, y$, and transitivity property, $x \, R_\gamma \, y$. Hence the result.

Corollary 2. Let $\mathcal{R}$ be a deterministic soft equivalence relation on $D = (F, \Gamma)$. For every fixed $\gamma \in \Gamma$ and $x, y \in F(\gamma)$, either of the following occurs:

$$[x]R_\gamma = [y]R_\gamma$$

or

$$[x]R_\gamma \cap [y]R_\gamma \neq \emptyset.$$

Proof. If the two equivalence classes have nonempty intersection, Theorem 10 implies that they are equal. Hence two distinct equivalence classes must be disjoint.

Remark 8. In contrast with an ordinary equivalence relation on U, a deterministic soft equivalence relation need not classify an object in the same way under different enriched deterministic parameters. Indeed, if $\gamma_1, \gamma_2 \in \Gamma$ and $x \in F(\gamma_1) \cap F(\gamma_2)$, it may happen that

$$[x]R_{\gamma_1} \neq [x]R_{\gamma_2}$$

Thus, equivalence classes in Deterministic Soft Set Theory are generally parameter-dependent.

Example 5. Let $U = \{c_1, c_2, c_3, c_4\}$ be a set of electoral candidates. Consider two enriched deterministic parameters

$$\gamma_1 = \text{eligible}^{+}_{\text{constitutional compliance}}, \qquad \gamma_2 = \text{eligible}^{+}_{\text{party clearance}}.$$

Suppose $F(\gamma_1) = F(\gamma_2) = \{c_1, c_2, c_3, c_4\}$. Define R_{γ_1} on $F(\gamma_1)$ and R_{γ_2} on $F(\gamma_2)$ by

$$\{\{c_1, c_2\}, \{c_3, c_4\}\} \quad \text{and} \quad \{\{c_1, c_3\}, \{c_2, c_4\}\},$$

respectively. Then both R_{γ_1} and R_{γ_2} are equivalence relations on their respective approximation sets. Therefore, $\mathcal{R} = \{R_{\gamma_1}, R_{\gamma_2}\}$ is a deterministic soft equivalence relation. Clearly,

$$[c_1]_{\mathcal{R}_{\gamma_1}} = \{c_1, c_2\}, \qquad [c_1]_{\mathcal{R}_{\gamma_2}} = \{c_1, c_3\}.$$

Hence,

$$[c_1]_{\mathcal{R}_{\gamma_1}} \neq [c_1]_{\mathcal{R}_{\gamma_2}}.$$

From Example 5 it is clear that the same candidate may belong to different equivalence classes depending on the deterministic condition under which equivalence is evaluated. This parameter-dependent classification is a feature that is not captured by an ordinary equivalence relation on U.

Let $D = (F, \Gamma)$ be a deterministic soft set over a universe U, where $\Gamma = A^{\#}_{\delta}B$ and

$$F: \Gamma \to \mathcal{P}(U).$$

A deterministic soft partition should partition each approximation set $F(\gamma)$ separately, since the approximation associated with one enriched deterministic parameter may differ from that associated with another.

Definition 3.12 (Deterministic Soft Partition). Let $D = (F, \Gamma)$ be a deterministic soft set. A deterministic soft partition of D is a family

$$\mathfrak{P} = \{\mathcal{P}_{\gamma}\}_{\gamma \in \Gamma},$$

where, for every $\gamma \in \Gamma$,

$$\mathcal{P}_{\gamma} = \{P_{\gamma_i} : i \in I_{\gamma}\}$$

is a partition of $F(\gamma)$. Thus, for every $\gamma \in \Gamma$,

i. $P_{\gamma_i} \neq \varnothing$ for every $i \in I_{\gamma}$,

ii. $F(\gamma) = \bigcup_{i \in I_{\gamma}} P_{\gamma_i}$,

iii. $P_{\gamma_i} \cap P_{\gamma_j} = \varnothing$ whenever $i \neq j$.

Remark 9. A deterministic soft partition is not, in general, a single partition of the whole universe U. Rather, it is an indexed family of partitions, one for each approximation set $F(\gamma)$. Consequently, the same element of U may occur in different blocks under different enriched deterministic parameters. This reflects the parameter-dependent nature of Deterministic Soft Set Theory.

Example 6. Let $U = \{c_1, c_2, c_3, c_4\}$, and let $\Gamma = \{\gamma_1, \gamma_2\}$, where

$$\gamma_1 = \text{eligible}^{+}_{\text{constitutional compliance}}, \qquad \gamma_2 = \text{eligible}^{+}_{\text{party clearance}}.$$

Suppose $F(\gamma_1) = F(\gamma_2) = \{c_1, c_2, c_3, c_4\}$. Define

$$\mathcal{P}_{\gamma_1} = \{\{c_1, c_2\}, \{c_3, c_4\}\},$$

and

$$\mathcal{P}_{\gamma_2} = \{\{c_1, c_3\}, \{c_2, c_4\}\}.$$

Then $\mathfrak{P} = \{\mathcal{P}_{\gamma_1}, \mathcal{P}_{\gamma_2}\}$ is a deterministic soft partition of D.

Theorem 11. Let $D = (F, \Gamma)$ be a deterministic soft set and $\mathcal{R} = \{R_\gamma\}_{\gamma\in\Gamma}$ be a deterministic soft equivalence relation on D. Then for every $\gamma \in \Gamma$,

$$\mathfrak{P}^{\mathcal{R}} = \{\mathcal{P}_\gamma^{\mathcal{R}}\}_{\gamma\in\Gamma}$$

is a deterministic soft partition of D, where

$$\mathcal{P}_\gamma^{\mathcal{R}} = \{[x]_{\mathcal{R},\gamma} : x \in F(\gamma)\}.$$

Proof. Suppose $\gamma \in \Gamma$ and R_γ is an equivalence relation on $F(\gamma)$. For every $x \in F(\gamma)$, it follows that $x \in [x]R_\gamma$ since R_γ is reflexive. Moreover,

$$F(\gamma) = \bigcup_{x\in F(\gamma)} [x]_{\mathcal{R},\gamma},$$

since every $x \in F(\gamma)$ is a member of its own equivalence class. Next, if $[x]R_\gamma \cap [y]R_\gamma \neq \emptyset$, then $[x]R_\gamma = [y]R_\gamma$. Hence distinct equivalence classes are disjoint. Therefore, $\mathcal{P}_\gamma^{\mathcal{R}}$ is a partition of $F(\gamma)$. Since γ was choosen arbitrary, it follows that $\mathfrak{P}_{\mathcal{R}} = \{\mathcal{P}_\gamma^{\mathcal{R}}\}_{\gamma\in\Gamma}$ is a deterministic soft partition of D.

Theorem 12. Let $D = (F, \Gamma)$ be a deterministic soft set, and $\mathfrak{P}^{\mathcal{R}} = \{\mathcal{P}_\gamma^{\mathcal{R}}\}_{\gamma\in\Gamma}$ be a deterministic soft partition of D. Then for every $\gamma \in \Gamma$,

$$\mathcal{R}^{\mathfrak{P}} = \{R_\gamma^{\mathfrak{P}}\}_{\gamma\in\Gamma}$$

is a deterministic soft equivalence relation on D, where $R_\gamma^{\mathfrak{P}} \subseteq F(\gamma) \times F(\gamma)$ is given by

$$x\, R_\gamma^{\mathfrak{P}}\, y \Leftrightarrow \text{there exists a block } P \in \mathcal{P}_\gamma \text{ such that } x, y \in P.$$

Proof. Suppose $\mathfrak{P} = \{\mathcal{P}_\gamma\}_{\gamma\in\Gamma}$ is a deterministic soft partition of D and $\gamma \in \Gamma$. Then we show that $R_\gamma^{\mathfrak{P}}$ is an equivalence relation on $F(\gamma)$.

Firstly, let $x \in F(\gamma)$. Since $\mathcal{P}_\gamma$ is a partition of $F(\gamma)$, there exists a unique block $P \in \mathcal{P}_\gamma$ such that $x \in P$. Hence $x\, R_\gamma^{\mathfrak{P}}\, x$, so reflexivity holds.

Secondly, let $x, y \in F(\gamma)$ suppose $x\, R_\gamma^{\mathfrak{P}}\, y$. Then there exists a block $P \in \mathcal{P}_\gamma$ such that $x, y \in P$. It follows that $y, x \in P$, so $y\, R_\gamma^{\mathfrak{P}}\, x$. Thus symmetry holds.

Lastly, suppose $x, y, z \in F(\gamma)$ and suppose $x\, R_\gamma^{\mathfrak{P}}\, y$ and $y\, R_\gamma^{\mathfrak{P}}\, z$ Then there exist blocks $P, Q \in \mathcal{P}_\gamma$ such that $x, y \in P$ and $y, z \in Q$. Since $y \in P \cap Q$, we have $P \cap Q \neq \emptyset$. Since $\mathcal{P}_\gamma$ is a partition, any two blocks are either identical or disjoint. Hence $P = Q$. Therefore, $x, z \in P$, This gives $x\, R_\gamma^{\mathfrak{P}}\, z$. Thus transitivity holds. Since γ was choosen arbitrarily, $\mathcal{R}_{\mathfrak{P}} = \{\mathcal{R}_\gamma^{\mathfrak{P}}\}_{\gamma\in\Gamma}$ is a deterministic soft equivalence relation on D.

4 Application in Electoral Candidate Selection

The preceding sections develop the structural foundations of Deterministic Soft Set Theory. In this section, we demonstrate the applicability of the framework to a candidate-selection problem arising in an electoral setting. The considered problem is as follows.

4.1 Illustrative Example

Suppose that a political party wishes to select one candidate from a finite set of aspirants to serve as its flag bearer. Let

$$U = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$$

denote the universe of seven candidates. Let

$$E = \{e_1, e_2, e_3, e_4\}$$

be the set of parameters, where

$e_1 = \mathit{Influence}$,

$e_2 = \mathit{Credibility}$,

$e_3 = \mathit{Public\ recognition}$, and

$e_4 = \mathit{Political\ strength}$.

Next let

$$B = \{b_1, b_2, b_3, b_4, b_5\}$$

be the set of deterministic factors, where

$b_1 = \mathit{Ethnic\ consideration}$,

$b_2 = \mathit{Religious\ consideration}$,

$b_3 = \mathit{Regional\ consideration}$,

$b_4 = \mathit{Past\ record}$, and

$b_5 = Educational\ background.$

The elements of B represent external deterministic conditions that may modify the manner in which the parameters are applied.

For every pair $(e_i, b_j) \in E \times B$, define deterministic parameter

$$d_{ij} = e_i^{\#} b_j, \qquad 1 \le i \le 4, \quad 1 \le j \le 5.$$

It follows that there are

$$|E||B| = 4 \times 5 = 20$$

deterministic evaluation conditions.

Let $F^{+}(d_{ij}) \subseteq U$ denote the set of candidates positively supported by the corresponding deterministic condition, and define

$$F^{-}(d_{ij}) = U \backslash F^{+}(d_{ij})$$

For each d_{ij}. The positive approximations used in this example are given by:

$F^{+}(d_{11}) = \{c_2, c_3, c_4, c_7\}$ $F^{+}(d_{12}) = \{c_1, c_2, c_6, c_7\}$
$F^{+}(d_{13}) = \{c_2, c_4, c_7\}$ $F^{+}(d_{14}) = \{c_1, c_2, c_4, c_5, c_7\}$
$F^{+}(d_{15}) = \{c_1, c_2, c_4, c_6\}$ $F^{+}(d_{21}) = \{c_2, c_6, c_7\}$
$F^{+}(d_{22}) = \{c_1, c_5, c_6, c_7\}$ $F^{+}(d_{23}) = \{c_1, c_2, c_3, c_5\}$
$F^{+}(d_{24}) = \{c_1, c_3, c_4, c_5\}$ $F^{+}(d_{25}) = \{c_6, c_7\}$
$F^{+}(d_{31}) = \{c_1, c_2\}$ $F^{+}(d_{32}) = \{c_4, c_5, c_6\}$
$F^{+}(d_{33}) = \{c_1, c_2, c_3, c_4, c_7\}$ $F^{+}(d_{34}) = \{c_2, c_4, c_7\}$
$F^{+}(d_{35}) = \{c_1, c_2, c_3, c_6\}$ $F^{+}(d_{41}) = \{c_1, c_2, c_5, c_6, c_7\}$
$F^{+}(d_{42}) = \{c_2, c_6\}$ $F^{+}(d_{43}) = \{c_1, c_2, c_4\}$
$F^{+}(d_{44}) = \{c_2, c_6, c_7\}$ $F^{+}(d_{45}) = \{c_2, c_7\}$

The corresonding negative approximations are determined uniquely by.

$$F^{-}(d_{ij}) = U \backslash F^{+}(d_{ij})$$

For each deterministic parameter d_{ij} and candidate c_k, define the positive binary value.

$$a_{ij,k} = \begin{cases} 1, & c_k \in F^{+}(d_{ij}), \\ 0, & c_k \notin F^{+}(d_{ij}). \end{cases}$$

and the negative binary value,

$$r_{ij,k} = \begin{cases} 1, & c_k \in F^{-}(d_{ij}), \\ 0, & c_k \notin F^{-}(d_{ij}). \end{cases}$$

Since,

$$F^{-}(d_{ij}) = U \backslash F^{+}(d_{ij})$$

We have,

$$r_{ij,k} = 1 - a_{ij,k}.$$

The resulting positive and negative binary matrices are given in Tables 1 and 2.

Table 1. Positive deterministic soft binary model for the electoral candidate-selection problem

$D^{\#}$	c_1	c_2	c_3	c_4	c_5	c_6	c_7
$d_{11}{}^{+}$	0	1	1	1	0	0	1
$d_{12}{}^{+}$	1	1	0	0	0	1	1
$d_{13}{}^{+}$	0	1	0	1	0	0	1
$d_{14}{}^{+}$	1	1	0	1	1	0	1
$d_{15}{}^{+}$	1	1	0	1	0	1	0
$d_{21}{}^{+}$	0	1	0	0	0	1	1
$d_{22}{}^{+}$	1	0	0	0	1	1	1
$d_{23}{}^{+}$	1	1	1	0	1	0	0
$d_{24}{}^{+}$	1	0	1	1	1	0	0
$d_{25}{}^{+}$	0	0	0	0	0	1	1
$d_{31}{}^{+}$	1	1	0	0	0	0	0
$d_{32}{}^{+}$	0	0	0	1	1	1	0
$d_{33}{}^{+}$	1	1	1	1	0	0	1
$d_{34}{}^{+}$	0	1	0	1	0	0	1
$d_{35}{}^{+}$	1	1	1	0	0	1	0
$d_{41}{}^{+}$	1	1	0	0	1	1	1
$d_{42}{}^{+}$	0	1	0	0	0	1	0
$d_{43}{}^{+}$	1	1	0	1	0	0	0
$d_{44}{}^{+}$	0	1	0	0	0	1	1
$d_{45}{}^{+}$	0	1	0	0	0	0	1
L_k	11	16	5	9	6	10	12

$D^{\#}$	c_1	c_2	c_3	c_4	c_5	c_6	c_7

Table 2. Negative deterministic soft binary model for the electoral candidate-selection problem

$D^{\#}$	c_1	c_2	c_3	c_4	c_5	c_6	c_7
d_{11}^-	1	0	0	0	1	1	0
d_{12}^-	0	0	1	1	1	0	0
d_{13}^-	1	0	1	0	1	1	0
d_{14}^-	0	0	1	0	0	1	0
d_{15}^-	0	0	1	0	1	0	1
d_{21}^-	1	0	1	1	1	0	0
d_{22}^-	0	1	1	1	0	0	0
d_{23}^-	0	0	0	1	0	1	1
d_{24}^-	0	1	0	0	0	1	1
d_{25}^-	1	1	1	1	1	0	0
d_{31}^-	0	0	1	1	1	1	1
d_{32}^-	1	1	1	0	0	0	1
d_{33}^-	0	0	0	0	1	1	0
d_{34}^-	1	0	1	0	1	1	0
d_{35}^-	0	0	0	1	1	0	1
d_{41}^-	0	0	1	1	0	0	0
d_{42}^-	1	0	1	1	1	0	1
d_{43}^-	0	0	1	0	1	1	1
d_{44}^-	1	0	1	1	1	0	0
d_{45}^-	1	0	1	1	1	1	0
K_k	9	4	15	11	14	10	8

For every candidate c_k, the total positive support is defined by

$$L_k = \sum_{i=1}^{4}\sum_{j=1}^{5} a_{ij,k},$$

and the total negative support is defined by

$$K_k = \sum_{i=1}^{4}\sum_{j=1}^{5} r_{ij,k}.$$

Since each candidate is evaluated under exactly twenty deterministic conditions,

$$L_k + K_k = 20.$$

We define the normalized deterministic electoral score of candidate c_k by

$$S(c_k) = \frac{L_k - K_k}{20} \times 100.$$

Since $K_k = 20 - L_k$, the score admits the equivalent representation

$$S(c_k) = \frac{2L_k - 20}{20} \times 100.$$

Therefore,

$$-100 \le S(c_k) \le 100.$$

A positive value indicates that the number of supporting deterministic conditions exceeds the number of opposing conditions, while a negative value indicates the reverse. $S(c_k)=0$ indicates a balance between positive and negative deterministic support.

Remark 10. The quantity $S(c_k)$ is a normalized deterministic support score and should not be interpreted as a probability of electoral victory or nomination.

4.2 Candidate-Selection Rule and Decision Algorithm

The preferred candidate is defined by

$$c^* \in \arg\max_{c_k \in U} S(c_k).$$

If the maximizer is unique, then c^* is selected as the preferred candidate. If two or more candidates attain the same maximal score, an additional tie-breaking rule must be specified.

Algorithm 1. Deterministic Soft Candidate-Selection Algorithm

1. Construct the enriched deterministic parameters $d_{ij}=e_i\#b_j$.
2. For every d_{ij} and $c_k \in U$, compute
$$a_{ij,k} = \begin{cases} 1, & c_k \in F^+(d_{ij}), \\ 0, & \text{otherwise.} \end{cases}$$
3. Compute
$$L_k = \sum_{i=1}^{4}\sum_{j=1}^{5} a_{ij,k}.$$
4. Compute
$$K_k = 20 - L_k.$$
5. Compute
$$S(c_k) = \frac{L_k - K_k}{20} \times 100.$$
6. Rank the candidates in decreasing order of $S(c_k)$.
7. Select
$$c^* \in \arg\max_{c_k \in U} S(c_k).$$

From the positive and negative binary matrices, the aggregate results are:

Table 3. Deterministic soft electoral scores

Candidate	L_k	K_k	$S(c_k)$
c_1	11	9	10%
c_2	16	4	60%
c_3	5	15	−50%
c_4	9	11	−10%
c_5	6	14	−40%
c_6	10	10	0%
c_7	12	8	20%

Consequently, $S(c_2) = 60$ is the largest deterministic support score. Hence,
$$c^* = c_2.$$

The resulting ranking is
$$c_2 \succ c_7 \succ c_1 \succ c_6 \succ c_4 \succ c_5 \succ c_3.$$

Candidate c_2 therefore emerges as the preferred alternative under the specified deterministic evaluation framework. The result illustrates an important feature of Deterministic Soft Set Theory. Candidate selection is not determined solely by the ordinary attributes in E. The same parameter is considered jointly with several deterministic factors in B, so that the final ranking reflects the combined effect of candidate attributes and external deterministic conditions.

Candidate c_2 receives positive support under sixteen of the twenty deterministic evaluation conditions and negative support under only four. Its resulting score, $S(c_2) = 60$, is therefore the highest among all alternatives. Candidate c_7 is second with $S(c_7) = 20$, followed by candidate c_1 with $S(c_1) = 10$. Candidate c_6 obtains a balanced score of $S(c_6) = 0$, whereas candidates c_4, c_5, and c_3 receive negative net deterministic support.

Accordingly, the model identifies c_2 as the preferred candidate under the specified parameter system.

5 Conclusion

This paper extends Deterministic Soft Set Theory by developing several structural concepts, including deterministic equality, inclusion, null and absolute deterministic soft sets, the NOT operation, deterministic soft equivalence relations, and deterministic soft partitions. A reduction criterion was also established to characterize when a deterministic soft set can be represented as an ordinary soft set.

The proposed framework was illustrated through an electoral candidate selection problem, where ordinary candidate attributes were evaluated together with deterministic external factors. The resulting model produced a normalized deterministic support score and identified c_2 as the preferred candidate under the specified decision environment.

These results strengthen the mathematical foundation of Deterministic Soft Set Theory and demonstrate its potential for structured decision-making problems. Future work may consider deterministic soft mappings, weighted models, quotient structures, lattice properties, and further applications in other decision environments.

References

[1] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20 (1), 87–96. https://doi.org/10.1016/S0165-0114(86) 80034-3

[2] Baldwin, J. F., & Guild, N. C. F. (1979). Comparison of fuzzy sets on the same decision space. Fuzzy Sets and Systems, 2 (3), 213–231. https://doi.org/10.1016/0165-0114(79)90003-9

[3] Chen, S. M., Hsiao, W. H., & Jong, W. T. (1997). Bidirectional ap- proximate reasoning based on interval-valued fuzzy sets. Fuzzy Sets and Systems, 91 (3), 339–353. https://doi.org/10.1016/S0165-0114(96) 00124-9

[4] Dubois, D., & Prade, H. (1985). A review of fuzzy set aggregation con- nectives. Information Sciences, 36 (1–2), 85–121. https://doi.org/10. 1016/0020-0255(85)90027-1

[5] Dubois, D., & Prade, H. (1989). Fuzzy sets, probability and mea- surement. European Journal of Operational Research, 40 (2), 135–154. https://doi.org/10.1016/0377-2217(89)90304-X

[6] Dubois, D., & Prade, H. (1991). Fuzzy sets in approximate reasoning, Part 1: Inference with possibility distributions. Fuzzy Sets and Systems, 40 (1), 143–202. https://doi.org/10.1016/0165-0114(91)90050-Z

[7] Gehrke, M., Walker, C., & Walker, E. A. (1996). Some com- ments on interval-valued fuzzy sets. International Journal of Intel- ligent Systems, 11 (10), 751–759. https://doi.org/10.1002/(SICI) 1098-111X(199610)11:10<751::AID-INT3>3.0.CO;2-G

[8] Grattan-Guinness, I. (1975). Fuzzy membership mapped onto interval and many-valued quantities. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 22, 149–160.

[9] Jahn, K. U. (1975). Intervall-wertige Mengen. Mathematische Nachrichten, 68 (1), 115–132. https://doi.org/10.1002/mana.

[10] Lee-Kwang, H., Song, Y. S., & Lee, K. M. (1994). Similarity measure be- tween fuzzy sets and between elements. Fuzzy Sets and Systems, 62 (3), 291–293. https://doi.org/10.1016/0165-0114(94)90113-9

[11] Molodtsov, D. (1999). Soft set theory—First results. Computers & Math- ematics with Applications, 37 (4–5), 19–31. https://doi.org/10.1016/ S0898-1221(99)00056-5

[12] Pawlak, Z. (1982). Rough sets. International Journal of Computer & Information Sciences, 11 (5), 341–356. https://doi.org/10.1007/ BF01001956

[13] Türksen, I. B. (1989). Four methods of approximate reasoning with interval-valued fuzzy sets. International Journal of Approximate Rea- soning, 3 (2), 121–142. https://doi.org/10.1016/0888-613X(89) 90030-2

[14] Türksen, I. B. (1992). Interval-valued fuzzy sets and compensatory AND. Fuzzy Sets and Systems, 51 (3), 295–307. https://doi.org/10. 1016/0165-0114(92)90210-U

[15] Türksen, I. B. (1994). Interval-valued fuzzy sets and fuzzy mea- sures. In Proceedings of the First International Joint Conference of NAFIPS/IFIS/NASA (pp. 317–321).

[16] Türksen, I. B. (1996). Non-specificity and interval-valued fuzzy sets. Fuzzy Sets and Systems, 80 (1), 87–100. https://doi.org/10.1016/ 0165-0114(95)00182-4

[17] Türksen, I. B., & Tian, Y. (1992). Interval-valued fuzzy sets rep- resentation on multiple antecedent fuzzy implications and reasoning.

Fuzzy Sets and Systems, 52 (2), 143–167. https://doi.org/10.1016/ 0165-0114(92)90085-R

[18] Türksen, I. B., & Zhong, Z. (1990). An approximate analogical reason- ing schema based on similarity measures and interval-valued fuzzy sets. Fuzzy Sets and Systems, 34 (3), 323–346. https://doi.org/10.1016/ 0165-0114(90)90235-M

[19] Xuecheng, L. (1992). Entropy, distance measure and similarity measure of fuzzy sets and their relations. Fuzzy Sets and Systems, 52 (3), 305–318. https://doi.org/10.1016/0165-0114(92)90239-Z

[20] Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8 (3), 338– 353. https://doi.org/10.1016/S0019-9958(65)90241-X

[21] Zadeh, L. A. (1978). Fuzzy sets as a basis for a theory of possibil- ity. Fuzzy Sets and Systems, 1 (1), 3–28. https://doi.org/10.1016/ 0165-0114(78)90029-5

[22] Alkhazaleh, S., Salleh, A. R., & Hassan, N. (2011). Possibility fuzzy soft set. Advances in Decision Sciences, 2011, Article 479756. https://doi.org/10.1155/2011/479756

[23] Alkhazaleh, S., Salleh, A. R., & Hassan, N. (2011). Fuzzy parameterized interval-valued fuzzy soft set. Applied Mathematical Sciences, 5 (67), 3335–3346.

[24] Alkhazaleh, S., & Salleh, A. R. (2012). Fuzzy soft multiset theory. Ab- stract and Applied Analysis, 2012, Article ID 350603.

[25] Alkhazaleh, S., & Salleh, A. R. (2011). Soft expert sets. Advances in Decision Sciences, 2011, Article 757868. https://doi.org/10.1155/2011/757868

[26] Alkhazaleh, S., & Salleh, A. R. (2014). Fuzzy soft expert set and its application. Applied Mathematics, 5, 1349–1368. https://doi.org/10.4236/am.2014.59127

[27] Atagün, A. O., Kamacı, H., & Oktay, O. (2018). Reduced soft matrices and generalized products with applications in decision making. Neural Computing and Applications, 29, 445–456.

[28] Çağman, N., & Enginoğlu, S. (2010). Soft matrix theory and its decision making. Computers & Mathematics with Applications, 59 (10), 3308–3314.

[29] Çağman, N., Çıtak, F., & Enginoğlu, S. (2010). Fuzzy parameterized fuzzy soft set theory and its applications. Turkish Journal of Fuzzy Sys- tems, 1 (1), 21–35.

[30] Çağman, N., Çıtak, F., & Enginoğlu, S. (2011). Fuzzy parameterized soft set theory and its applications. Annals of Fuzzy Mathematics and Informatics, 2 (2), 219–226.

[31] Chen, D., Tsang, E. C. C., & Yeung, D. S. (2003). Some notes on the pa- rameterization reduction of soft sets. In Proceedings of the International Conference on Machine Learning and Cybernetics (pp. 1442–1445).

[32] Chen, D., Tsang, E. C. C., & Wang, X. (2005). The parameterization reduction of soft sets and its applications. Computers & Mathematics with Applications, 49, 757–763.

[33] Debnath, S. (2021). Fuzzy hypersoft sets and its weightage operator for decision making. Journal of Fuzzy Extension and Applications, 2 (2), 163–170.

[34] Elijah, E. E., & Muhammad, U. F. (2021). On full fuzzy parameterized soft set. International Journal of Mathematical Sciences and Optimiza- tion: Theory and Applications, 7 (2), 76–84.

[35] Feng, F., Li, Y., & Çağman, N. (2012). Generalized uni-int decision making schemes based on choice value soft sets. European Journal of Operational Research, 220 (1), 162–170.

[36] Feng, Q., & Zhou, Y. (2014). Soft discernibility matrix and its applica- tions in decision making. Applied Soft Computing, 24, 749–756.

[37] George, U. S., Elijah, E. E., Ibrahim, M., & Yakubu, A. (2025). De- terministic soft set with application in decision making. Science Forum (Journal of Pure and Applied Mathematics), 25 (3), 1–21.

[38] Herawan, T., & Deris, M. M. (2010). Soft decision making for patients suspected influenza. In Computational Science and Its Applications (pp. 622–631).

[39] Herawan, T., & Deris, M. M. (2011). A soft set approach for association rules mining. Knowledge-Based Systems, 24 (1), 186–195.

[40] Inthumathi, V., Chitra, V., & Jayasree, S. (2017). The role of opera- tors on soft set in decision making problems. International Journal of Computational and Applied Mathematics, 12, 899–910.

[41] Kamacı, H., Saltık, K., Akız, H. F., & Atagün, A. O. (2018). Cardinality inverse soft matrix theory and its applications in multicriteria group decision making. Journal of Intelligent & Fuzzy Systems, 34, 2031–2049.

[42] Kharal, A. (2014). Soft approximations and uni-int decision making. The Scientific World Journal, 2014, Article 327408.

[43] Maji, P. K., Biswas, R., & Roy, A. R. (2001). Fuzzy soft sets. Journal of Fuzzy Mathematics, 9, 589–602.

[44] Maji, P. K., Biswas, R., & Roy, A. R. (2001). Intuitionistic fuzzy soft sets. Journal of Fuzzy Mathematics, 9, 677–692.

[45] Maji, P. K., Roy, A. R., & Biswas, R. (2002). An application of soft sets in a decision making problem. Computers & Mathematics with Applications, 44, 1077–1083.

[46] Petchimuthu, S., Garg, H., Kamacı, H., & Atagün, A. O. (2020). The mean operators and generalized products of fuzzy soft matrices and their applications in MCGDM. Computational and Applied Mathematics, 39, Article 262.