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Intuitionistic δ^* -Open Sets in Intuitionistic Topological Spaces

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Abstract

This paper introduces a fresh direction of open sets in intuitionistic topological spaces (ITS), described as intuitionistic δ^* -open sets. The proposed class is described by using the intuitionistic δ -interior operator together with the ordinary intuitionistic interior and closure operators. The main aim is to construct a class that contains intuitionistic δ -open sets and gives a flexible structure for studying generalized interior, closure, exterior, boundary, and limit points. The paper presents the basic definitions of intuitionistic sets (IOS) and ITSs, defines the proposed class, proves its fundamental properties, and gives finite examples showing that the fresh direction is wider than the classical intuitionistic open class. The results show that the family of intuitionistic δ^* -open sets is stable under arbitrary unions and supports well-defined derived operators. These operators provide a systematic framework for future extensions of continuity, compactness, and separation axioms in ITS.

Keywords

ITS;
IOS;
 δ -open set;
 δ^* -open set;
Intuitionistic Interior;
Intuitionistic Closure;
Limit Point.



1 Introduction

The theory of IOSs provides a suitable mathematical framework for studying objects that cannot be described only by membership or non-membership. In this setting, each set is represented by two components. The first component describes the degree or part of membership, while the second component describes the degree or part of non-membership. This structure was developed from the fundamental work of Atanassov on intuitionistic fuzzy sets, where the presence of an indeterminacy part gives the model a wider ability to describe uncertain information than ordinary set theory and classical fuzzy set theory [1].

Because of this feature, intuitionistic structures have become useful in many branches of mathematics and applied sciences. Topology is one of the important fields in which intuitionistic ideas have produced strong extensions. Coker introduced ITSs and showed how open sets, closed sets, interior, closure, and related operators can be studied under intuitionistic membership and non-membership components [2]. Later, Bayhan and Çoker studied separation axioms in ITSs and clarified how classical T_1 and T_2 -type ideas can be reformulated in the IOSting [3]. These studies established a solid base for developing fresh directions of open sets and new topological operators in intuitionistic spaces. Generalized open sets play a central role in topology because they allow researchers to describe weaker and more flexible forms of openness. In the IOSting, this role becomes more important because each topological relation must respect both the membership part and the non-membership part. Several studies have considered different kinds of intuitionistic fuzzy open sets, intuitionistic fuzzy interiors, and related operators [4].

Other works extended the theory through intuitionistic fuzzy Top-sp.s, smooth Top-sp.s, and

regular subspace structures [5-7]. These developments show that the study of generalized openness remains an active direction in ITS. Recent works by Atanassov introduced new topological and modal topological operators over intuitionistic fuzzy sets [8,9]. These contributions indicate that ITS still needs new operator-based classes that can connect classical topological concepts with more flexible intuitionistic structures. Further developments in intuitionistic fuzzy modal multi-topological structures also show that generalized intuitionistic topological models can support richer mathematical systems [10]. At the applied level, intuitionistic topological and fuzzy-topological tools have been used in classification, clustering, GIS-related topological rules, and neural-network environments [11-13]. This confirms that new intuitionistic open classes are not only theoretically useful but may also provide future tools for applied models involving uncertainty.

This paper proposes a fresh direction of generalized open sets in ITSs. The proposed class is described as the class of intuitionistic δ^* -open sets. The construction depends on three operators: the intuitionistic interior operator, the intuitionistic closure operator, and the intuitionistic δ -interior operator. The proposed class is designed to contain every intuitionistic δ -open set, while it may contain sets that are not ordinary IOSs.

The paper is organized as follows. Section 2 presents the basic definitions of IOSs, ITSs, and the standard operators. Section 3 introduces the proposed δ^* -open set and studies its basic properties. Section 4 develops the associated δ^* -interior, δ^* -closure, δ^* -exterior, δ^* -boundary, and δ^* -derived set. Section 5 gives illustrative examples. Section 6 states the main conclusions.

2. Preliminaries

Throughout this paper, X denotes a non-empty set. An IOS on X is written as

$$K = \langle X, K_1, K_2 \rangle,$$

where $K_1 \subseteq X$, $K_2 \subseteq X$, and $K_1 \cap K_2 = \emptyset$.

The set A_1 is described as the set of members of A , while A_2 is described as the set of non-members of A . The remaining part

$$X - (K_1 \cup K_2)$$

represents the indeterminacy region of K .

Definition 2.1. [1] Let $K = \langle X, K_1, K_2 \rangle$ and $B = \langle X, B_1, B_2 \rangle$ be two IOSs on X . We say that K is contained in B , and write $K \subseteq B$, if $K_1 \subseteq B_1$ and $K_2 \supseteq B_2$.

Thus, inclusion increases the membership part and decreases the non-membership part.

Definition 2.2. [1] Two IOSs K and B on X are equal if $K_1 = B_1$ and $K_2 = B_2$. Equivalently, $K = B \Leftrightarrow K \subseteq B$ and $B \subseteq K$.

Definition 2.3. [1] The complement of an IOS $K = \langle X, K_1, K_2 \rangle$ is described by $K^c = \langle X, K_2, K_1 \rangle$. Hence $(K^c)^c = K$.

Definition 2.4. [1] For two IOSs $K = \langle X, K_1, K_2 \rangle$ and $B = \langle X, B_1, B_2 \rangle$, the union and intersection are described by $K \cup B = \langle X, K_1 \cup B_1, K_2 \cap B_2 \rangle$, and $K \cap B = \langle X, K_1 \cap B_1, K_2 \cup B_2 \rangle$.

For an arbitrary family $\{K_i : i \in \Lambda\}$ of IOSs, where $K_i = \langle X, K_{i1}, K_{i2} \rangle$, we define

$$\bigcup_{i \in \Lambda} K_i = \left\langle X, \bigcup_{i \in \Lambda} K_{i1}, \bigcap_{i \in \Lambda} K_{i2} \right\rangle,$$

and

$$\bigcap_{i \in \Lambda} K_i = \left\langle X, \bigcap_{i \in \Lambda} K_{i1}, \bigcup_{i \in \Lambda} K_{i2} \right\rangle.$$

Definition 2.5. [1] The null IOS and the whole IOS are respectively described by $0_{\sim} = \langle X, \emptyset, X \rangle$, and $1_{\sim} = \langle X, X, \emptyset \rangle$.

Definition 2.6. [1] A family τ of IOSs on X is described as an ITS on X if Certain circumstances are met::

1. $0_{\sim} \in \tau$ and $1_{\sim} \in \tau$.
2. If $G, H \in \tau$, then $G \cap H \in \tau$.
3. If $\{G_i : i \in \Lambda\} \subseteq \tau$, then $\bigcup_{i \in \Lambda} G_i \in \tau$.

The pair (X, τ) is described as an ITS. Every member of τ is described as an IOS. The complement of an IOS is described as an intuitionistic closed set.

Definition 2.7. [2] Let K be an IOS in (X, τ) . The intuitionistic interior of K is described by $\text{Int}_I(K) = \bigcup \{G : G \in \tau \text{ and } G \subseteq K\}$.

The intuitionistic closure of K is described by $\text{Clu}_I(K) = \bigcap \{F : F^c \in \tau \text{ and } K \subseteq F\}$. The following standard relations hold: $\text{Int}_I(K) \subseteq K \subseteq \text{Clu}_I(K)$, $K \subseteq B \Rightarrow \text{Int}_I(K) \subseteq \text{Int}_I(B)$, and $K \subseteq B \Rightarrow \text{Clu}_I(K) \subseteq \text{Clu}_I(B)$.

Definition 2.8. [2] An IOS K in (X, τ) is described as intuitionistic regular open if $K = \text{Int}_I(\text{Clu}_I(K))$. The family of all intuitionistic regular open sets in (X, τ) is symbolised by $\mathcal{RO}_I(X, \tau)$.

Definition 2.9. [4] Let K be an IOS in (X, τ) . The intuitionistic δ -interior of K is described by $\text{Int}_{I\delta}(K) = \bigcup \{R : R \in \mathcal{RO}_I(X, \tau) \text{ and } R \subseteq K\}$. An IOS K is described as intuitionistic δ -open if $K = \text{Int}_{I\delta}(K)$.

3. The Intuitionistic δ^* -Open Set

This section introduces the proposed class and proves its basic structural properties.

Definition 3.1. Let (X, τ) be an ITS. An IOS K in X is described as an intuitionistic δ^* -open set (δ^* -op-s.) if

$$K \subseteq \text{Int}_I(\text{Clu}_I(\text{Int}_{I\delta}(K))).$$

The family of all δ^* -op-s.s in (X, τ) is symbolised by $\mathcal{O}_{I\delta^*}(X, \tau)$. The complement of an δ^* -op-s. is described as an intuitionistic δ^* -closed set. The family of all intuitionistic δ^* -closed sets is symbolised by $\mathcal{C}_{I\delta^*}(X, \tau)$.

Theorem 3.2. In every ITS (X, τ) , $0_\sim, 1_\sim \in \mathcal{O}_{I\delta^*}(X, \tau)$.

Proof: For $0_\sim$, we have $\text{Int}_{I\delta}(0_\sim) = 0_\sim$.

Therefore $\text{Int}_I(\text{Clu}_I(\text{Int}_{I\delta}(0_\sim))) = \text{Int}_I(\text{Clu}_I(0_\sim))$.

Since $0_\sim \subseteq \text{Int}_I(\text{Clu}_I(0_\sim))$, it follows that $0_\sim$ is δ^* -op-s.. For $1_\sim$, since $1_\sim$ is regular open, we obtain $\text{Int}_{I\delta}(1_\sim) = 1_\sim$. Thus

$$\text{Int}_I(\text{Clu}_I(\text{Int}_{I\delta}(1_\sim))) = \text{Int}_I(\text{Clu}_I(1_\sim)) = 1_\sim.$$

Hence $1_\sim$ is δ^* -op-s.. This completes the proof.

Theorem 3.3. If K is intuitionistic δ -open, then K is δ^* -op-s..

Proof: Assume that K is δ -open. Then $K = \text{Int}_{I\delta}(K)$.

By the basic inclusion property of the interior and closure operators, we have $K \subseteq \text{Clu}_I(K)$. Since K is a union of intuitionistic regular open sets, each point of K lies in some intuitionistic regular open set R . Specifically, $R \subseteq K$. For every such R , we have

$$R = \text{Int}_I(\text{Clu}_I(R)).$$

Because $R \subseteq K$, monotonicity of the closure gives

$$\text{Clu}_I(R) \subseteq \text{Clu}_I(K).$$

By monotonicity of the interior, we get

$$R = \text{Int}_I(\text{Clu}_I(R)) \subseteq \text{Int}_I(\text{Clu}_I(K)).$$

Taking the union over all such regular open sets R , we obtain

$$K \subseteq \text{Int}_I(\text{Clu}_I(K)).$$

Since $K = \text{Int}_{I\delta}(K)$, this becomes

$$K \subseteq \text{Int}_I(\text{Clu}_I(\text{Int}_{I\delta}(K))).$$

Thus K is δ^* -op-s..

Corollary 3.4. Every intuitionistic regular open set is δ^* -op-s..

Proof: Every intuitionistic regular open set is intuitionistic δ -open by Definition 2.9. Hence the result follows directly from Theorem 3.3.

Theorem 3.5. The arbitrary union of δ^* -op-s.s is δ^* -op-s.. That is, if $K_i \in \mathcal{O}_{I\delta^*}(X, \tau)$ for every $i \in \Lambda$, then

$$\bigcup_{i \in \Lambda} K_i \in \mathcal{O}_{I\delta^*}(X, \tau)$$

Proof: Let $K = \bigcup_{i \in \Lambda} K_i$. Since each K_i is δ_I^* -op-s., we have $K_i \subseteq \text{Int}_I(\text{Clu}_I(\text{Int}_{I\delta}(K_i)))$ for every $i \in \Lambda$. By monotonicity of $\text{Int}_{I\delta}$, we have

$$K_i \subseteq K \Rightarrow \text{Int}_{I\delta}(K_i) \subseteq \text{Int}_{I\delta}(K).$$

Hence $\text{Int}_I(\text{Clu}_I(\text{Int}_{I\delta}(K_i))) \subseteq \text{Int}_I(\text{Clu}_I(\text{Int}_{I\delta}(K)))$.

Therefore $K_i \subseteq \text{Int}_I(\text{Clu}_I(\text{Int}_{I\delta}(K)))$

for every $i \in \Lambda$. Taking the union gives $\bigcup_{i \in \Lambda} K_i \subseteq \text{Int}_I(\text{Clu}_I(\text{Int}_{I\delta}(K)))$.

Since $K = \bigcup_{i \in \Lambda} K_i$, we obtain $K \subseteq \text{Int}_I(\text{Clu}_I(\text{Int}_{I\delta}(K)))$.

Thus K is δ_I^* -op-s..

Theorem 3.6. An IOS F is intuitionistic δ^* -closed if and only if F^c is δ_I^* -op-s..

Proof: This follows immediately from the definition of intuitionistic δ^* -closed sets as complements of δ_I^* -op-s.

Thus $F \in \mathcal{C}_{I\delta^*}(X, \tau) \Leftrightarrow F^c \in \mathcal{O}_{I\delta^*}(X, \tau)$.

This proves the result.

4. Operators Generated by Intuitionistic δ^* -Open Sets

The previous section shows that the proposed class is closed under arbitrary unions. Hence it produces natural generalized operators.

Definition 4.1. Let K be an IOS in (X, τ) . The intuitionistic δ^* -interior of K is described by $\text{Int}_{I\delta^*}(K) = \bigcup \{G : G \in \mathcal{O}_{I\delta^*}(X, \tau) \text{ and } G \subseteq K\}$.

Theorem 4.2. For any IOSs K and B in (X, τ) , the following statements hold: $\text{Int}_{I\delta^*}(K) \subseteq K, K \subseteq B \Rightarrow \text{Int}_{I\delta^*}(K) \subseteq \text{Int}_{I\delta^*}(B)$, and $K \in \mathcal{O}_{I\delta^*}(X, \tau) \Leftrightarrow K = \text{Int}_{I\delta^*}(K)$.

Proof: The first relation follows from the definition because $\text{Int}_{I\delta^*}(K)$ is the union of all δ_I^* -op-s. subsets of K . For the second relation, assume $K \subseteq B$. Every δ_I^* -op-s. contained in K is also contained in B . Therefore, the union of all δ_I^* -op-s.s of K is contained in the union of all δ_I^* -op-s.s of B . Hence

$$\text{Int}_{I\delta^*}(K) \subseteq \text{Int}_{I\delta^*}(B).$$

For the third relation, suppose that K is δ_I^* -op-s.. Since $K \subseteq K$, the set K appears in the family used to define $\text{Int}_{I\delta^*}(K)$. Thus $K \subseteq \text{Int}_{I\delta^*}(K)$.

Together with the first relation, this gives $K = \text{Int}_{I\delta^*}(K)$.

Conversely, if $K = \text{Int}_{I\delta^*}(K)$, then K is a union of δ_I^* -op-s.s. By Theorem 3.5, this union is δ_I^* -op-s.. Hence K is δ_I^* -op-s..

Definition 4.3. Let K be an IOS in (X, τ) . The intuitionistic δ^* -closure of K is described by $\text{Clu}_{I\delta^*}(K) = \bigcap \{F : F \in \mathcal{C}_{I\delta^*}(X, \tau) \text{ and } K \subseteq F\}$.

Equivalently, $\text{Clu}_{I\delta^*}(K) = (\text{Int}_{I\delta^*}(K^c))^c$.

Theorem 4.4. For any IOSs K and B in (X, τ) , the following statements hold: $K \subseteq \text{Clu}_{I\delta^*}(K), K \subseteq B \Rightarrow \text{Clu}_{I\delta^*}(K) \subseteq \text{Clu}_{I\delta^*}(B)$, and $K \in \mathcal{C}_{I\delta^*}(X, \tau) \Leftrightarrow K = \text{Clu}_{I\delta^*}(K)$.

Proof: The first relation follows because $\text{Clu}_{I\delta^*}(K)$ is the intersection of all intuitionistic δ^* -closed sets containing K . For the second relation, assume $K \subseteq B$. Every intuitionistic δ^* -closed set containing B also contains K . Hence the family used to define $\text{Clu}_{I\delta^*}(B)$ is a subfamily of the family used to define $\text{Clu}_{I\delta^*}(K)$. Therefore,

$$\text{Clu}_{I\delta^*}(K) \subseteq \text{Clu}_{I\delta^*}(B).$$

For the third relation, if K is intuitionistic δ^* -closed, then K belongs to the family of intuitionistic δ^* -closed supersets of itself. Hence $\text{Clu}_{I\delta^*}(K) \subseteq K$.

Together with the first relation, we obtain $K = \text{Clu}_{I\delta^*}(K)$.

Conversely, if the equality holds, then K is the intersection of intuitionistic δ^* -closed sets. Since complements of arbitrary unions of intuitionistic δ^* -open sets are intersections of intuitionistic δ^* -closed sets, K is intuitionistic δ^* -closed.

Definition 4.5. The intuitionistic δ^* -exterior of K is described by $\text{Ext}_{I\delta^*}(K) = \text{Int}_{I\delta^*}(K^c)$.

Theorem 4.6. For every IOS K , $\text{Ext}_{I\delta^*}(K) = (\text{Clu}_{I\delta^*}(K))^c$.

Proof: By Definition 4.3, $\text{Clu}_{I\delta^*}(K) = (\text{Int}_{I\delta^*}(K^c))^c$. Taking complements on both sides gives

$$(\text{Clu}_{I\delta^*}(K))^c = \text{Int}_{I\delta^*}(K^c).$$

Using Definition 4.5, we get $(\text{Clu}_{I\delta^*}(K))^c = \text{Ext}_{I\delta^*}(K)$.

This proves the theorem.

Definition 4.7. The intuitionistic δ^* -boundary of K is described by $\text{Bd}_{I\delta^*}(K) = \text{Clu}_{I\delta^*}(K) \cap \text{Clu}_{I\delta^*}(K^c)$.

Equivalently, $\text{Bd}_{I\delta^*}(K) = \text{Clu}_{I\delta^*}(K) - \text{Int}_{I\delta^*}(K)$,

where the difference is understood by $K - B = K \cap B^c$.

Theorem 4.8. For every IOS K in (X, τ) , $1_{\sim} = \text{Int}_{I\delta^*}(K) \cup \text{Bd}_{I\delta^*}(K) \cup \text{Ext}_{I\delta^*}(K)$.

Moreover, $\text{Int}_{I\delta^*}(K) \cap \text{Ext}_{I\delta^*}(K) = 0_{\sim}$.

Proof: By Definition 4.5, $\text{Ext}_{I\delta^*}(K) = \text{Int}_{I\delta^*}(K^c)$.

Also, by Definition 4.7, $\text{Bd}_{I\delta^*}(K) = \text{Clu}_{I\delta^*}(K) \cap \text{Clu}_{I\delta^*}(K^c)$.

Since $\text{Clu}_{I\delta^*}(K) = (\text{Ext}_{I\delta^*}(K))^c$, and $\text{Clu}_{I\delta^*}(K^c) = (\text{Int}_{I\delta^*}(K))^c$, the elements of the space are divided into three possible regions: the δ^* -interior of K , the δ^* -exterior of K , and the part that belongs to neither of them. This remaining part is exactly the δ^* -boundary. Hence

$$1_{\sim} = \text{Int}_{I\delta^*}(K) \cup \text{Bd}_{I\delta^*}(K) \cup \text{Ext}_{I\delta^*}(K).$$

The second relation follows because $\text{Int}_{I\delta^*}(K)$ is contained in K , while $\text{Ext}_{I\delta^*}(K)$ is contained in K^c . Therefore their intersection is $0_{\sim}$.

Definition 4.9. Let K be an IOS in (X, τ) . A point $x \in X$ is described as an intuitionistic δ^* -limit point of K if every intuitionistic δ^* -open neighbourhood G of x satisfies $(G - \{x\}_\sim) \cap K \neq 0_\sim$. The set of all intuitionistic δ^* -limit points of K is described as the intuitionistic δ^* -derived set of K and is symbolised by $\text{Der}_{I\delta^*}(K)$. Here $\{x\}_\sim$ denotes the intuitionistic singleton generated by x , namely $\{x\}_\sim = \langle X, \{x\}, X - \{x\} \rangle$.

Theorem 4.10. For every IOS K , $\text{Der}_{I\delta^*}(K) \subseteq \text{Clu}_{I\delta^*}(K)$.

Proof: Let $x \in \text{Der}_{I\delta^*}(K)$. Assume, for contradiction, that $x \notin \text{Clu}_{I\delta^*}(K)$. Then x belongs to the complement of $\text{Clu}_{I\delta^*}(K)$, which is the δ^* -exterior of K . Hence it has an δ^* -op-s. neighborhood G of x . Specifically, $G \cap K = 0_\sim$.

This implies $(G - \{x\}_\sim) \cap K = 0_\sim$, which contradicts the assumption that x is a δ^* -limit point of K . Therefore, $x \in \text{Clu}_{I\delta^*}(K)$.

Thus $\text{Der}_{I\delta^*}(K) \subseteq \text{Clu}_{I\delta^*}(K)$.

This completes the proof.

Theorem 4.11. If (X, τ) is an intuitionistic T_1 space, then for every IOS K , $\text{Clu}_{I\delta^*}(K) = K \cup \text{Der}_{I\delta^*}(K)$.

Proof: First, since $K \subseteq \text{Clu}_{I\delta^*}(K)$ and by Theorem 4.10 $\text{Der}_{I\delta^*}(K) \subseteq \text{Clu}_{I\delta^*}(K)$, we get $K \cup \text{Der}_{I\delta^*}(K) \subseteq \text{Clu}_{I\delta^*}(K)$.

Conversely, let $x \in \text{Clu}_{I\delta^*}(K)$ and assume that $x \notin K$.

Let G be any intuitionistic δ^* -open neighbourhood of x . Since x belongs to the δ^* -closure of K , every such neighbourhood must meet K . Therefore, $G \cap K \neq 0_\sim$.

Because $x \notin K$, the intersection point cannot be only x . Hence $(G - \{x\}_\sim) \cap K \neq 0_\sim$.

Thus x is an intuitionistic δ^* -limit point of K . Therefore $x \in \text{Der}_{I\delta^*}(K)$.

This gives $\text{Clu}_{I\delta^*}(K) \subseteq K \cup \text{Der}_{I\delta^*}(K)$.

The two inclusions prove the equality.

5. Examples

Example 5.1. Let $X = \{a, b, c\}$. Define the following IOSs: $0_\sim = \langle X, \emptyset, X \rangle$, $1_\sim = \langle X, X, \emptyset \rangle$, $G = \langle X, \{a\}, \{c\} \rangle$, and $H = \langle X, \{a, b\}, \emptyset \rangle$.

Consider $\tau = \{0_\sim, G, H, 1_\sim\}$. Then τ is an ITS on X . Indeed, $G \cap H = \langle X, \{a\}, \{c\} \rangle = G$, and $G \cup H = \langle X, \{a, b\}, \emptyset \rangle = H$. All arbitrary unions of members of τ remain in τ , and finite intersections remain in τ .

Compute $\text{Clu}_I(G)$. The closed sets are the complements of members of τ : $0_\sim^c = 1_\sim$, $1_\sim^c = 0_\sim$, $G^c = \langle X, \{c\}, \{a\} \rangle$, and $H^c = \langle X, \emptyset, \{a, b\} \rangle$. The smallest closed IOS containing G is $1_\sim$. Hence $\text{Clu}_I(G) = 1_\sim$.

Therefore, $\text{Int}_I(\text{Clu}_I(G)) = \text{Int}_I(1_\sim) = 1_\sim$. Since $G \neq 1_\sim$, we conclude that G is not intuitionistic regular open.

On the other hand, $\text{Int}_I(\text{Clu}_I(1_\sim)) = 1_\sim$, so $1_\sim$ is intuitionistic regular open. Also, $0_\sim$ is regular open. Hence $\mathcal{RO}_I(X, \tau) = \{0_\sim, 1_\sim\}$.

Example 5.2. Let $X = \mathbb{R}^2$ with the ordinary Euclidean topology. Embed every ordinary subset $E \subseteq X$ as the IOS $E_\sim = \langle X, E, X - E \rangle$.

Let D be the open unit disc and let L be the horizontal diameter of D , where $D = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 < 1\}$, and $L = \{(x, 0) \in \mathbb{R}^2: -1 < x < 1\}$.

Define $U = D - L$. The set U is the union of the upper and lower open half-discs. Each half-disc is regular open in the induced ITS, and hence $\text{Int}_{I\delta}(U_{\sim}) = U_{\sim}$.

Moreover, $\text{Int}_I(\text{Clu}_I(U_{\sim})) = D_{\sim}$. Now define $K = U \cup \{(0, 0)\}$.

Then $U_{\sim} \subseteq K_{\sim} \subseteq D_{\sim}$.

Therefore, $K_{\sim} \subseteq \text{Int}_I(\text{Clu}_I(\text{Int}_{I\delta}(K_{\sim})))$.

Thus $K_{\sim}$ is intuitionistic δ^* -open. However, K is not open in the ordinary Euclidean topology because the point $(0, 0)$ does not have an open disc contained in K . Hence $K_{\sim}$ is intuitionistic δ^* -open but not ordinary intuitionistic open. This example proves that the proposed class may be strictly larger than the ordinary intuitionistic open class.

Example 5.3. Let $K_{\sim}$ be the IOS constructed in Example 5.2. Since $K_{\sim}$ is intuitionistic δ^* -open, we have $\text{Int}_{I\delta^*}(K_{\sim}) = K_{\sim}$.

The intuitionistic δ^* -exterior is $\text{Ext}_{I\delta^*}(K_{\sim}) = \text{Int}_{I\delta^*}(K_{\sim}^c)$.

The intuitionistic δ^* -boundary is $\text{Bd}_{I\delta^*}(K_{\sim}) = \text{Clu}_{I\delta^*}(K_{\sim}) \cap \text{Clu}_{I\delta^*}(K_{\sim}^c)$.

A point $x \in X$ belongs to the intuitionistic δ^* -derived set of $K_{\sim}$ if every intuitionistic δ^* -open neighbourhood of x meets $K_{\sim}$ at a point different from x . In symbols, $x \in \text{Der}_{I\delta^*}(K_{\sim}) \Leftrightarrow (G - \{x\}_{\sim}) \cap K_{\sim} \neq 0_{\sim}$ for every $G \in \mathcal{O}_{I\delta^*}(X, \tau)$ with $x \in G$.

Conclusions

This paper introduced a fresh direction of generalized open sets in ITSSs, described as intuitionistic δ^* -open sets. The class was described by the inclusion

$$K \subseteq \text{Int}_I(\text{Clu}_I(\text{Int}_{I\delta}(K))).$$

The proposed definition uses the interaction between the δ -interior, ordinary closure, and ordinary interior. This gives a meaningful extension of intuitionistic δ -open sets and provides a suitable structure for building generalized operators.

The main results show that every intuitionistic δ -open set is intuitionistic δ^* -open and that arbitrary unions of intuitionistic δ^* -open sets remain intuitionistic δ^* -open. Based on this stability property, the paper defined and studied the intuitionistic δ^* -interior, δ^* -closure, δ^* -exterior, δ^* -boundary, and δ^* -derived set.

Several identities were proved, including the exterior-closure relation and the relation between closure and derived sets under the intuitionistic T_1 condition.

Future work can develop intuitionistic δ^* -continuous functions, intuitionistic δ^* -compactness, and intuitionistic δ^* -separation axioms. These extensions may support new applications in mathematical models where membership and non-membership information must be treated together.

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