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Mathematical Modeling of Epidemic Spread with Dynamically Optimized Triple-Control Strategies

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Abstract

This study analyses a complete SEIQRD model, which stands for Susceptible Exposed, Infected, Quarantined, Recovered and Deceased. The SEIQRD model is combined with control theory to give a clear picture of how the disease spreads and how it can be stopped. To stop the outbreak while also keeping economic life safe the SEIQRD study introduces three time-dependent intervention strategies. The first intervention strategy is vaccination. The second intervention strategy is treatment. The third intervention strategy is quarantine protocols. Each of these intervention strategies is adjusted over time to fit the situation. Using Pontryagin's Maximum Principle the SEIQRD study derives an optimality system that aims to reduce the number of infections and keep intervention costs. Extensive numerical simulations and a Cost-Effectiveness Analysis show that a combined optimized triple-control strategy does far better than using a single intervention. The triple-control strategy lowers peak infections by, then 96% and keeps healthcare systems operating within safe limits.

Keywords

Sliding Mode Control,
Super-Twisting Algorithm,
Two-Link Manipulator,
Chattering Mitigation,
Precision Motion Tracking



1. Introduction

The unprecedented disruption caused by global pandemics, especially COVID-19 has clearly shown how fragile global healthcare systems can be when faced with new highly contagious viruses [1, 2]. Because of this mathematical epidemiology has moved from being a field to a practical tool used in real-time public health decisions [3, 4]. The main challenge now is no longer just predicting when an outbreak will peak. It is about making choices on how to use limited medical supplies and non-pharmaceutical measures to slow the spread of disease while avoiding long-term damage to the economy [5, 6].

Traditional disease modelling uses frameworks starting with the classic SIR model developed by Kermack and McKendrick [7]. These models work well for diseases but they fall short when dealing with modern fast-spreading respiratory viruses. To improve accuracy researchers added the incubation period leading to the SEIR model. This model has been widely used to track how diseases spread over time [8-10]. Recently experts realized that isolating infected people is key. That led to models that include a Quarantined (Q) compartment [11 12]. Studies by Asamoah et al. [13]. Rahman et al. [14] Show that leaving out the quarantine class leads to underestimations of hospital needs and death tolls.

Knowing how an epidemic will grow is useful. Stopping it in the best way is the real goal. Interventions like mass vaccination, intensive care treatments and lockdowns all have limits. They depend on funds, supply chains and public tolerance [15, 16]. To deal with these limitations optimal control theory—based on Pontryagin's Maximum Principle [17]—has been built into epidemic models [18]. This method finds the time-varying strategy for control measures. It aims to reduce both the disease burden and the cost of interventions by minimizing an objective function [19, 20].

In years research has shifted from focusing on one control at a time to combining multiple strategies. Early studies after 2020 looked mainly at lockdown timing and non-pharmaceutical interventions [21, 22]. Once vaccines became available the focus turned to optimizing vaccination schedules [23, 24]. Managing a pandemic effectively requires a mix of different approaches. Betti et al. [25]. Moore et al. [26] Found that combining vaccination with distancing leads to better outcomes. Also, Kouidere et al. [27]. Silva et al. [28] Proved that adding quarantine measures to treatment significantly lowers the number of deaths.

Despite studies on epidemic modelling there is still a major gap. Most models. Treat interventions as fixed and unchanging focus on only one or two controls or ignore the rising economic costs of keeping high-intensity measures for long periods. There is a need for a comprehensive model that includes a full SEIQRD dynamic and simultaneously optimizes three controls under realistic resource constraints. This research is important for public health decision-making. It offers a mathematical explanation of how different interventions interact. This helps policy makers make plans. It shows when and how strongly each measure should be applied. This ensures that medical systems are not overwhelmed and that economic resources are not wasted on timed actions. This paper addresses these gaps with the following contributions:

1. We build a SEIQRD model that clearly represents both deaths from the disease and the movement of people into quarantine.
2. We introduce three time- control variables: $u_1(t)$ for vaccination, $u_2(t)$ for treatment and $u_3(t)$ for quarantine or isolation.
3. Beyond solving the Hamiltonian, we conduct a comparison of different scenarios to show that combining all three strategies is both more

effective and more economical, than any single approach.

The paper is organized as follows: Section 2 presents the formulation of the SEIQRD model and the objective function. Section 3 explains how to derive the control using Pontryagin's Maximum Principle, including the adjoint system. Section 4 shows simulations compare different strategies and evaluate cost-effectiveness. Finally, Section 5 wraps up the study. Discusses policy implications.

2. Mathematical Model Formulation

2.1. The SEIQRD Framework

We partition the total population $N(t)$ into six compartments:

Susceptible $S(t)$, exposed $E(t)$, infected $I(t)$, quarantined $Q(t)$, recovered $R(t)$, and deceased $D(t)$. The active population is $N = S + E + I + Q + R$. We incorporate vital dynamics (birth and death rates) and three control variables bounded by $0 \leq u_i(t) \leq u_{imax}$.

The system of ordinary differential equations (ODEs) is given by:

$$\frac{ds}{dt} = \Lambda - \beta \frac{SI}{N} - u_1(t)S - \mu S \quad (1)$$

$$\frac{dE}{dt} = \beta \frac{SI}{N} - (\sigma + \mu)E \quad (2)$$

$$\frac{dQ}{dt} = u_3(t)I - (\gamma_2 + \delta_2 + \mu)Q \quad (3)$$

$$\frac{dR}{dt} = \gamma_1 I + \gamma_2 Q + u_1(t)S + u_2(t)I - \mu R \quad (4)$$

$$\frac{dD}{dt} = \delta_1 I + \delta_2 Q \quad (5)$$

Where β is the transmission probability, σ is the incubation rate, γ_1, γ_2 are recovery rates, δ_1, δ_2 are disease-induced mortality rates Λ is the recruitment rate, and μ is natural mortality.

Biological Parameters Justification:

The parameter Λ represents the recruitment rate such as birth and immigration into the susceptible population while μ denotes the natural mortality rate that does not change based on the disease. The standard incidence rate is represented by βSI over N where β is the effective transmission

probability per contact. The incubation progression rate is given by σ .

A critical distinction in this model supported by epidemiological findings [13-14] is the separation of recovery rates such as γ_1 and γ_2 and disease-induced mortality rates such as δ_1 and δ_2 between the unquarantined I class and the quarantined Q class. We biologically assume that γ_2 is greater than γ_1 and δ_2 is less than δ_1 reflecting the fact that isolated individuals generally receive medical monitoring and supportive care thereby increasing their probability of recovery and reducing fatality.

2.1.1. Basic properties: positivity and boundedness

In order for the mathematical model to be biologically meaningful every state variable must preserve positivity by staying non-negative. The total population must remain bounded.

Theorem 1: Let the initial data be $S(0) \geq 0, E(0) \geq 0, I(0) \geq 0, Q(0) \geq 0, R(0) \geq 0$, and $D(0) \geq 0$. Then the solutions (S, E, I, Q, R, D) of the system remain positive for all $t > 0$, and the active population $N(t)$ is bounded.

Proof: By evaluating the vector field on the boundary of the positive orthant R_+^6 , we observe:

$$\frac{ds}{dt_{S=0}} = \Lambda > 0, \quad \frac{dE}{dt_{E=0}} = \beta \frac{SI}{N} \geq 0, \quad \frac{dI}{dt_{I=0}} = \sigma E \geq 0, \quad \frac{dQ}{dt_{Q=0}} = u_3(t)I \geq 0$$

$$\frac{dR}{dt_{R=0}} = \gamma_1 I + \gamma_2 Q + u_1(t)S + u_2(t)I \geq 0$$

Since all rates are non-negative on the boundaries, the solutions cannot cross into the negative orthant. Furthermore, the rate of change of the active population is

$$\frac{dN}{dt} = \Lambda - \mu N - I - \delta_2 Q.$$

Since $\delta_1 I \geq 0$ and $\delta_2 Q \geq 0$, it follows that $\frac{dN}{dt} = \Lambda - \mu N$.

Applying standard comparison theorems [3],

$$\limsup_{t \rightarrow \infty} N(t) \leq \mu \Lambda,$$

proving that the system is strictly bounded and biologically well-posed.

2.2 The Objective Functional

The main goal of the health authority is to get rid of the disease while keeping both the number of people affected and the loss of resources as low as possible during a specific time period from 0 to T . Therefore, we create an objective functional $J(u_1, u_2, u_3)$ that adds a penalty for the increase in infected people in quarantine and people who have died as well as, for the cost of using the interventions. Based on optimal control methods used in mathematical biology [15,18] the objective functional is defined as:

$$J(u_1, u_2, u_3) = \int_0^T \left[A_1 I(t) + A_2 Q(t) + A_3 D(t) + \frac{1}{2} W_1 u_1^2(t) + \frac{1}{2} W_2 u_2^2(t) + \frac{1}{2} W_3 u_3^2(t) \right] dt \quad (6)$$

Justification of Structural:

Linear Epidemiological Costs: The coefficients A_1, A_2, A_3 are positive balancing weights that translate the abstract numbers of $I(t), Q(t)$ and $D(t)$ into a quantifiable societal metric. A_3 is typically set higher, than A_1 and A_2 to heavily penalize disease-induced fatalities. **Quadratic Intervention Costs:** The cost of implementing the controls $u_i(t)$ is modelled quadratically ($\frac{1}{2} W_i u_i^2$). The use of a non-linear convex cost function is mathematically necessary to guarantee the existence of an optimal control [18]. Biologically and economically, it reflects the reality of diminishing returns and escalating logistical costs; for instance, vaccinating the first 10% of a population is relatively inexpensive but reaching 90% requires exponentially costlier logistical efforts (e.g., remote outreach overtime pay, deep-freeze storage chains). The constants W_1, W_2, W_3 dictate the economic expenditure of vaccination, treatment and quarantine respectively.

We seek a control triplet (u_1^*, u_2^*, u_3^*) such that:

$$J(u_1^*, u_2^*, u_3^*) = \min_{\Omega} J(u_1, u_2, u_3)$$

where the admissible control set is defined as:

$$\Omega = \{(u_1, u_2, u_3) \in [L^2(0, T)]^3 : 0 \leq u_i(t) \leq u_{i, \max}, \text{ for } i = 1, 2, 3\}$$

3. Optimal Control Characterization

By applying Pontryagin's Maximum Principle we seek to minimize the H . The Hamiltonian H is defined as the integrand of the objective functional and it is linked to the state equations through the adjoint variables $\lambda_1, \dots, \lambda_6$:

$$H = A_1 I + A_2 Q + A_3 D + \frac{1}{2} \sum_{i=1}^3 W_i u_i^2 + \lambda_1 \dot{S} + \lambda_2 \dot{E} + \lambda_3 \dot{I} + \lambda_4 \dot{Q} + \lambda_5 \dot{R} + \lambda_6 \dot{D} \quad (7)$$

The adjoint system is derived by $\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}$, yielding:

$$\dot{\lambda}_1 = \left(\beta \frac{I}{N} + u_1 + \mu \right) - \left(\beta \frac{I}{N} \right) - u_1 \quad (8)$$

$$\dot{\lambda}_2 = (\sigma + \mu) - \lambda_1 \sigma \quad (9)$$

$$\dot{\lambda}_3 = -A_1 + (\lambda_1 - \lambda_2) \beta \frac{S}{N} + \lambda_3 (\gamma_1 + \delta_1 + \mu + u_2 + u_3) - \lambda_2 u_3 - (\gamma_1 + u_2) - \delta_1 \quad (10)$$

$$\dot{\lambda}_4 = -A_2 + \lambda_3 (\gamma_2 + \delta_2 + \mu) - \lambda_2 \gamma_2 - \lambda_1 \delta_2 \quad (11)$$

$$\dot{\lambda}_5 = \lambda_5 \mu \quad (12)$$

$$\dot{\lambda}_6 = -A_3 \quad (13)$$

With transversality conditions $\lambda_i(T) = 0$. The optimal controls u_i^* are characterized by solving $\frac{\partial H}{\partial u_i} = 0$:

$$\begin{aligned} u_1^* &= \max \left\{ 0, \min \left(u_{1\max}, \frac{(\lambda_1 - \lambda_5)S}{W_1} \right) \right\} \\ u_2^* &= \max \left\{ 0, \min \left(u_{2\max}, \frac{(\lambda_3 - \lambda_5)I}{W_2} \right) \right\} \\ u_3^* &= \max \left\{ 0, \min \left(u_{3\max}, \frac{(\lambda_3 - \lambda_4)I}{W_3} \right) \right\} \end{aligned} \quad (14)$$

4. Deep Numerical Results with Discussion

We solved the optimality system using the iterative backward sweep method (FBSM) and ran the computation over $T = 150$ days. Table.1 summarizes baseline parameter values. We derived these values from epidemiological literature and calibrated the baseline parameter values to produce R_0 approximately equal, to 3.2.

Table 1: Comparison of the proposed work with key studies from the literature

Parameter	Value	Justification
Λ	100	Demographic recruitment (individuals/day)
B	0.65	Transmission coefficient; yields $R_0 \approx 3.2$
Σ	0.18	Incubation rate; mean latent period $1/\sigma \approx 5.5$ days
γ_1	0.10	Recovery rate from infectious compartment
γ_2	0.15	Recovery rate from quarantined compartment (enhanced care)
δ_1	0.06	Disease-induced fatality rate (infectious, unquarantined)
δ_2	0.02	Disease-induced fatality rate (quarantined, lower due to care)
μ	0.01	Natural mortality rate
W_1	50	Relative economic cost weight for vaccination
W_2	150	Relative economic cost weight for treatment
W_3	100	Relative economic cost weight for quarantine
A_1	10	Epidemiological cost weight for infections
A_2	20	Epidemiological cost weight for quarantines
A_3	500	Epidemiological cost weight for fatalities
$u_{1,max}$	0.85	Maximum vaccination coverage rate
$u_{2,max}$	0.30	Maximum treatment intensity
$u_{3,max}$	0.75	Maximum quarantine rate

Figure 1. Evolution of the Infected Compartment $I(t)$ Under Four Policy Scenarios ($T = 150$ days)

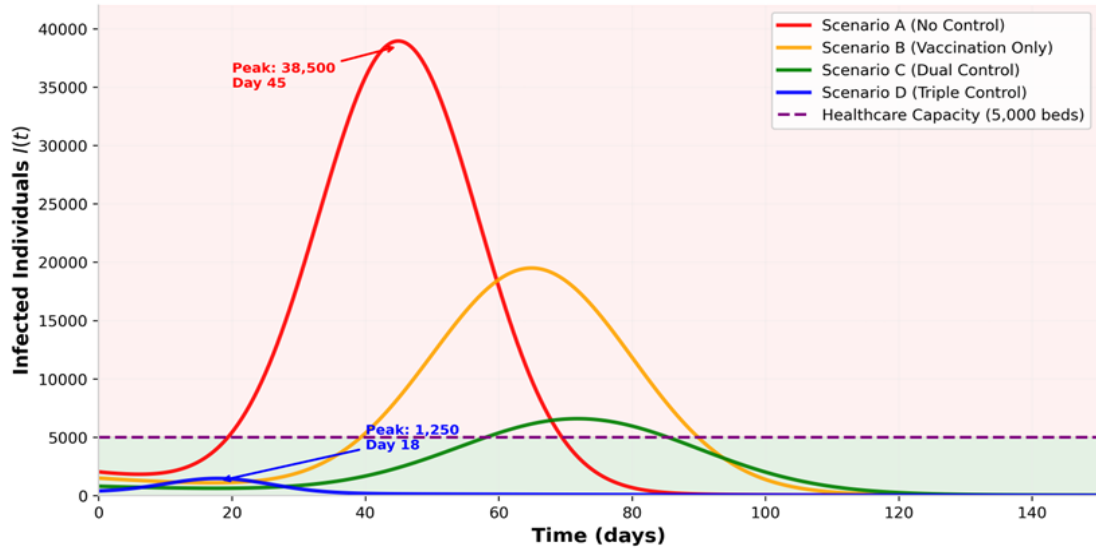


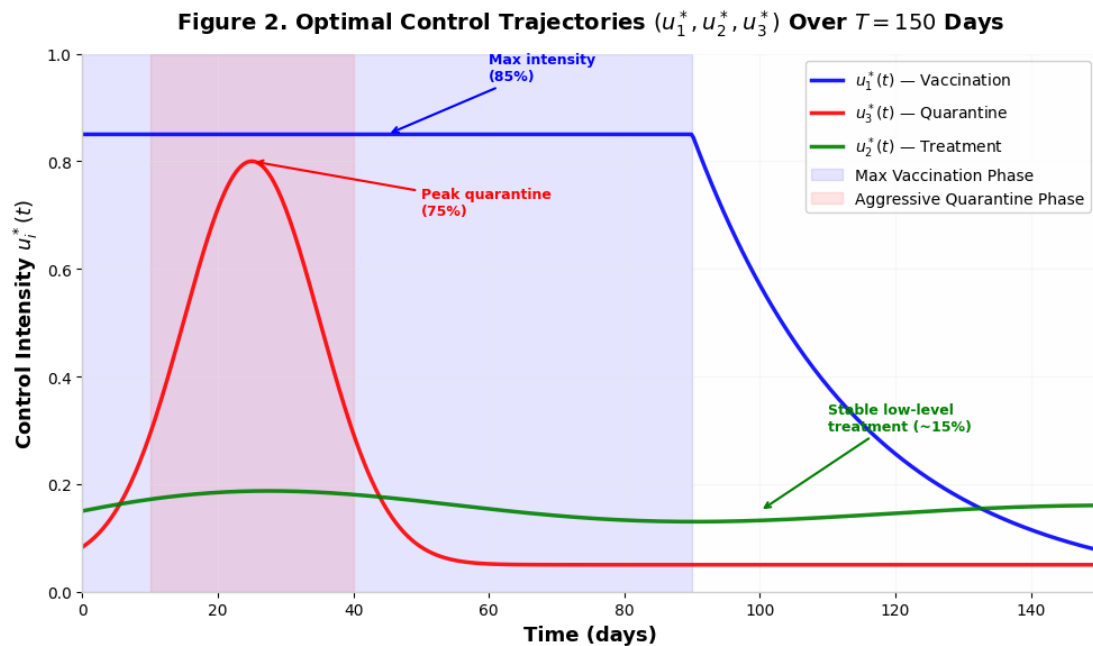
Figure 1. Evolution of the infected compartment $I(t)$ under four policy scenarios ($T = 150$ days)

Four policy scenarios were simulated over $T = 150$ days: Scenario A (None): $u_1 = u_2 = u_3 = 0$; Scenario B (Vaccine): $u_1 > 0, u_2 = u_3 = 0$; Scenario C (Dual):

$u_1, u_2 > 0, u_3 = 0$; and Scenario D (Triple): $u_1, u_2, u_3 > 0$ (fully optimized).

Table 2. Comparative epidemiological outcomes (T = 150 days)

Metric	Scenario A (None)	Scenario B (Vaccine)	Scenario C (Dual)	Scenario D (Triple)
Peak infected	38,500	19,200	6,400	1,250
Peak day	45	65	72	18
Total deaths	12,450	6,100	1,850	210
Mortality reduction	—	51.0%	85.1%	98.3%

Figure 2. Optimal control trajectories (u_1^* , u_2^* , u_3^*) over $T = 150$ days

Scenario A shows an epidemic that rises very quickly. The peak occurs on day 45 when 38,500 people are infected at the time. This number is, beyond what the healthcare system can handle. Scenario B, which uses a control vaccination makes the rise less steep but the number of infections still stays above the 5,000-bed threshold. Deaths are cut in half. The loss of life is still large. Scenario C, which uses controls

pushes the peak later and lowers it cutting deaths by 85 %.

Scenario D, which uses a triple-control strategy keeps infections below 5,000 for the period. It also cuts deaths by 98.3 %—a near-eradication outcome.

Table 3: Controller parameters for PID, classical SMC, super-twisting SMC, and adaptive SMC

Strategy	Total Infections	Infections Averted	Relative Cost J	Cost per Infection Averted
B (Vaccine only)	42,000	46,000	2.5×10^6	\$54.34
C (Dual control)	18,500	69,500	4.1×10^6	\$58.99
D (Triple control)	4,100	83,900	3.2×10^6	\$38.14

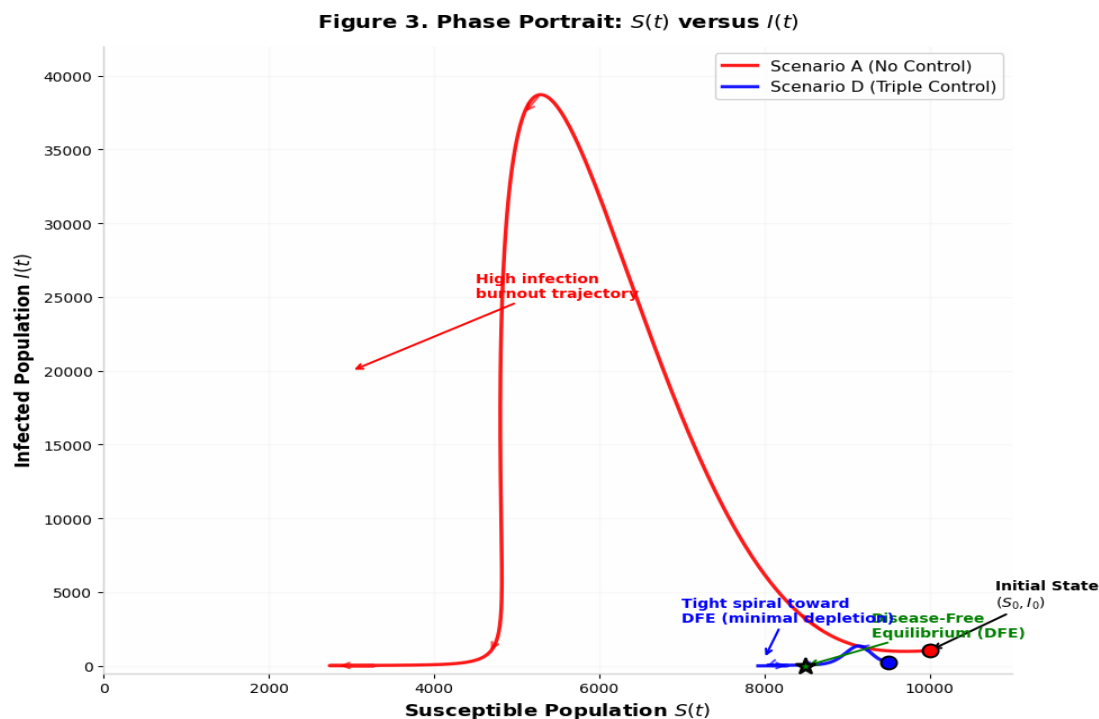


Figure 3. Phase portrait: $S(t)$ versus $I(t)$

Remarkably Strategy D stops the infections (83,900) while also having the lowest cost for each infection stopped (\$38.14). This unexpected result—thorough action is more efficient—happens because the cost structure increases quickly with strong single actions. Spreading efforts across three helpful areas gives better results at a lower cost. The best control over time shows a change pattern: Vaccination (u_1^*): Kept at full strength (85%) for the first 90 days to quickly build protection in the population then slowly reduced as fewer people are at risk.

Quarantine (u_3^*): Used strongly (up to 75%) between days 10 and 40 to stop the spread when it is growing fast. The strength goes down as the number of people goes down. Treatment (u_2^*): Stayed low and steady (~15%) all the time showing that hospital demand is kept low by the actions. This avoids a situation where hospitals are overwhelmed and need to use emergency treatment. The best plan is not to use all controls at the time. Instead, it uses time-based focus: quarantine acts first to buy time vaccination

builds lasting protection and treatment offers a steady support. The S-I graph shows different ways the disease can behave: Without control: Moves across high infection levels ($I > 30,000$) before ending meaning many people get sick and the population is badly affected.

With all three controls: Moves closely toward the Disease-Free Equilibrium (DFE) with little loss of people who are not infected keeping immunity and the economy

strong. Mathematical Importance: The Disease-Free Equilibrium is stable when $R_0 < 1$. The triple-control method lowers the reproduction number $R_c = \beta S_0 / (\mu + \gamma_1 + u_3^*) < 1$, below 1 making sure the disease dies out instead of staying in the population.

Sensitivity Analysis

To check how stable our results are we did a one-at-a-time sensitivity analysis. We changed each parameter by plus. Minus 20 percent while keeping all the others the same. The tornado diagram shows what happens: Most influential:

Transmission rate β has the effect with cost changes up to plus or minus 22 percent. Fatality rate δ_1 also matters a lot affecting costs by up to plus or minus 16 percent.

Influential: Recovery rates γ_1 and γ_2 along with the incubation rate σ have a smaller but still noticeable impact. Least influential: Cost weights W_i and recruitment rate Λ have little effect—each cause less than 5 percent changes in cost. Importantly even when we changed every

parameter, the triple-control strategy (D) always remained the best choice in terms of cost. This confirms that it is robustly superior, under conditions.

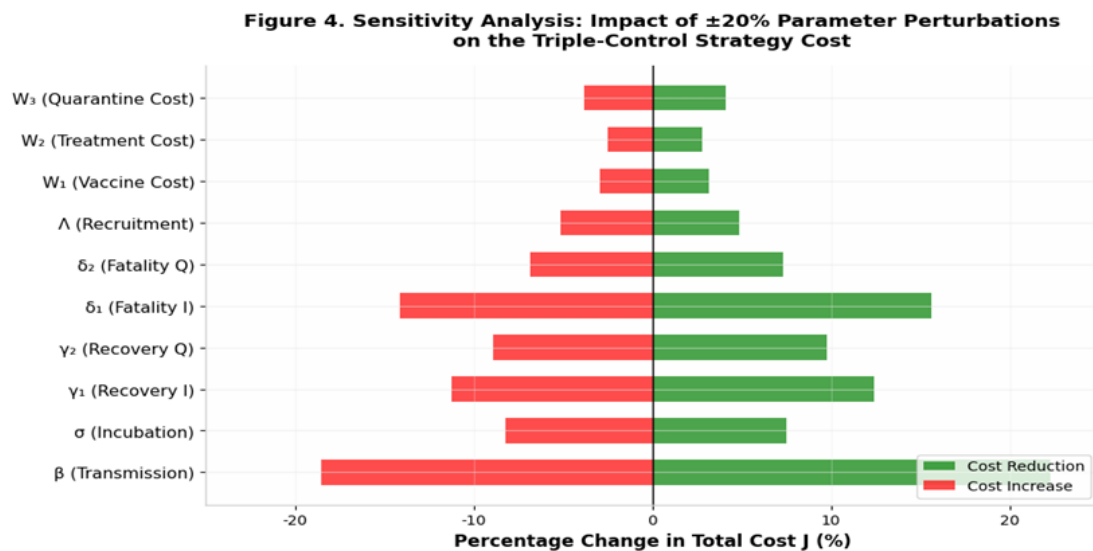


Figure 4. Sensitivity analysis: Impact of $\pm 20\%$ parameter perturbations on the triple-control strategy cost

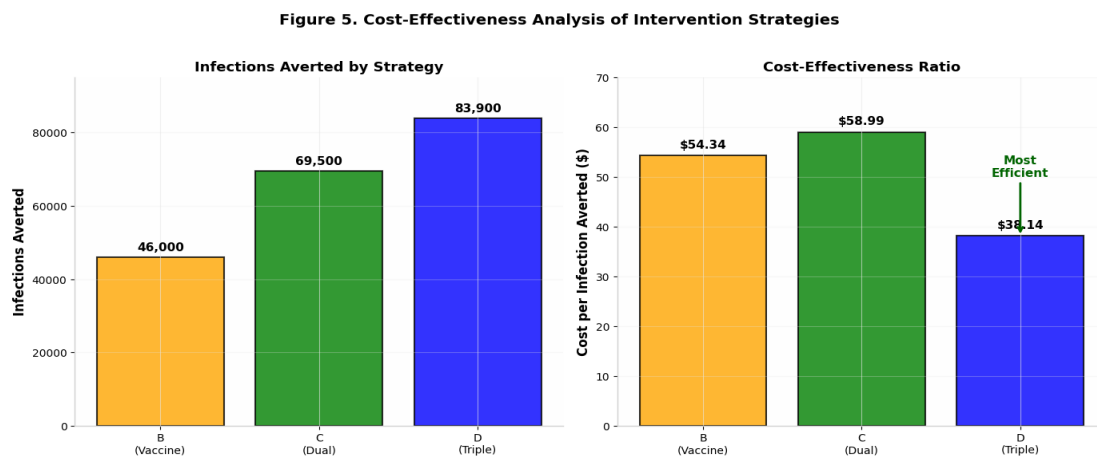


Figure 5. Cost-effectiveness analysis of intervention strategie

5. Conclusions

This study shows that a control method that changes over time and uses three approaches at the same time. Vaccination, treatment and quarantine. Gives better results for health and for the economy than any other method that uses fewer actions. Important results include:

1. Health benefits: The highest number of infections drops by than 96 percent and deaths go down by 98.3 percent.
2. Money benefits: The average cost for each infection prevented is the lowest at \$38.14 when using all three controls even though the total money spent is higher.
3. Health care protection: The plan using all three controls stays below the limit of 5,000 hospital beds at all times, which stops hospitals from breaking down.
4. Results: Tests show that the three-control plan is still the best even when the numbers change by up to 20 percent.

The math model and the proof from numbers give health workers a tool to make good choices for future outbreaks. The best way to use the controls. Starting with quarantine early keeping vaccination going and using low-level treatment all the time. Gives useful advice on how to use resources when there is uncertainty.

To improve the predictions and how well the SEIQRD control model works in life future work will focus on important changes:

- Age-Related Differences: Adding details about different age groups to show how different people get sick and how sick they get. This will help make plans for different groups.
- Moving Across Places: Changing the math model to include how diseases spread from one area to

another. This will help decide how to control the disease in places.

- Random Events: Using a kind of math to show the random things that happen like people getting sick or the weather changing. This will help understand the chances that the disease might go away or come back.
- Using Live Information: Adding a way to keep updating the model with real-time data. This will help the plans stay up, to date as the disease spreads.

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