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A Study of the Stability of the Numerical Method of RKM-Type for Solving Ordinary Differential Equations of Fifth-Order

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Abstract

Numerical method of RKM-type for solving the ordinary differential equations (ODEs) of fifth-order plays an important role in the accurate and efficient solutions of mathematical models arising in engineering and science. The stability is one of the most important properties of numerical methods for solving ODEs. In this paper, the stability properties of the RKM numerical method for solving ODEs of fifth-order has been introduced and investigated. The stability analysis of this method, polynomial of the stability and the region of absolute stability have been emphasized. However, a comprehensive stability analysis is performed by applying the proposed method to an appropriate linear test problem. The corresponding amplification factor and stability polynomial are derived, and the absolute stability condition is established based on the requirement that the modulus of the amplification factor does not exceed unity. The region of absolute stability is then determined and graphically illustrated in the complex plane. Furthermore, the effects of the step size and problem parameters on the stability of the numerical solution are examined. Numerical experiments are presented to verify the theoretical results and to demonstrate the behavior of the proposed method for representative fifth-order ODEs. The results show that the proposed method produces stable and reliable numerical approximations when the step size lies within the derived stability restrictions.

Keywords

Stability,
RK,
RKM,
RKN,
RKD,
Zero Stability,
Absolute Stability.



1. Introduction

Differential equations (DEs) have a significance rule in engineering and science. They are used in many applications whenever a scientific quantity changes with time, space, or another variable. The scientific fields of applications of DEs are in the physics like the principle motion, waves, heat, electricity, while in the principle of chemistry are reaction rates and chemical kinetics and in the biology are population growth and disease models. In addition to agriculture like soil-water movement, and crop growth [1-17]. Moreover, in the environmental science like pollution, transport, dispersion. In the same time, in geology like meteorology, lastly, in astronomy like atmospheric and climate models. Also, DEs are central to engineering because physical systems are governed by relationships between forces, flows, energy, temperature, pressure, and rates of change.

Stability is one of the most important properties of a numerical method for solving ODEs. It determines whether small errors introduced during numerical computation remain controlled or grow to unacceptable values. Every numerical calculation contains errors. Stability determines whether these errors are damped, controlled, or amplified. The stability of the numerical methods is a fundamental issue due to inappropriate choices of the uncontrolled step size, which may lead to numerical oscillations, instability, significant errors or divergent numerical solutions in the computed solution[18-24]. A comprehensive stability analysis is presented by applying the RKM method to the linear test problem associated with fifth-order ODEs.

The stability is controlled propagation of numerical errors. A numerical method is called stable if small perturbations in the initial data, intermediate numerical values, round-off errors, or truncation errors do not grow uncontrollably as the computation proceeds. In general, For the fifth-order RKM methods in the uploaded paper, the problem is treated directly rather than first converting it to five first-order equations. The paper emphasizes this direct formulation as a computational advantage.

Many authors derived some numerical methods for solving different orders of ODEs [25], for special case, [26] has derived some numerical methods for solving third- order ODEs. However, Runge-Kutta-Mohamed (RKM) family is designed to solve fifth-order initial-value problems (IVPs) directly, rather than first reducing them to two first-order equations [27-29]. This makes RKM method particularly attractive when the problem has the form $y^{(5)}(x) = f(x, y(x))$, especially for mechanical and oscillatory systems. Stability analysis of RKM methods is commonly performed by applying the method to a linear second-order test equation and examining the eigenvalues of the resulting two-by-two stability matrix. This is the natural analogue of the scalar stability function used for ordinary Runge-Kutta methods. See, for example, the literature on absolute-stability regions of explicit RKN methods.

2. Background

In general, the IVPs of ODEs of fifth-order has the following formula:

$$f\left(x, y(x), y'(x), y''(x), y'''(x), y^{(4)}(x), y^{(5)}(x)\right)=0, \quad x_0 \leq x \leq x_n \quad (1)$$

with the initial conditions (ICs)

$$y(x_0) = \alpha_i, \quad i = 0, 1, 2, 3, 4 \quad (2)$$

The IVP of special quasi-linear ODEs of fifth-order has the following form:

$$y^{(5)}(x) = f(x, y(x)) = 0, \quad x_0 \leq x \leq x_n \quad (3)$$

For solving IVP in (1)-(2) or (3)-(2), there are some classical or modern analytical numerical methods. In some cases of these problems, the analytical methods are not done or not sufficient to find the suitable solutions for them due to the complexity of these problems. Usually, for these cases, the researchers used the numerical methods, especially the RK-type numerical methods, for solving these IVPs. [27] derived RKM method for solving quasi-linear ODEs of fifth-order. The general forms of this method of RKM type is written as follows:

$$z_{i+1} = z_i + h z_i' + \frac{h^2}{2!} z_i'' + \frac{h^3}{3!} z_i''' + \frac{h^4}{4!} z_i^{(4)} + h^5 \sum_{j=1}^s b_j \kappa_j, \quad (4)$$

$$z_{i+1}' = z_i' + h z_i'' + \frac{h^2}{2!} z_i''' + \frac{h^3}{3!} z_i^{(4)} + h^4 \sum_{j=1}^s b_j' \kappa_j, \quad (5)$$

$$z_{i+1}'' = z_i'' + h z_i''' + \frac{h^2}{2!} z_i^{(4)} + h^3 \sum_{j=1}^s b_j'' \kappa_j, \quad (6)$$

$$z_{i+1}^{(3)} = z_i^{(3)} + h z_i^{(4)} + h^2 \sum_{j=1}^s b_j^{(3)} \kappa_j, \quad (7)$$

$$z_{i+1}^{(4)} = z_i^{(4)} + h \sum_{j=1}^s b_j^{(4)} \kappa_j, \quad (8)$$

where,

$$\kappa_i = f(x_i, z_i), \quad i = 0, 1, 2, 3, \dots, n-1 \quad (9)$$

and

$$\kappa_j = f \left(x_i + c_j h, z_i + c_j h z_i' + c_j^2 \frac{h^2}{2!} z_i'' + c_j^3 \frac{h^3}{3!} z_i^{(3)} + c_j^4 \frac{h^4}{4!} z_i^{(4)} + h^5 \sum_{j=1}^{s-1} b_j \kappa_j \right), \quad j = 2, 3, \dots \quad (10)$$

where b_j, c_j, k_j and a_{ij} are the RKM-methods coefficients for m and $k = 2, 3, \dots, s$. RKM is an explicit-method if $a_{ij} = 0$ for $j \geq i$ and otherwise the RKM method is an implicit. RKM coefficients can be expressed in the following table using Butcher notation:

Table 1: The Butcher Tableau for RKM Method

C	A
	b^T
	b'^T
	b''^T
	b'''^T
	$b^{(4)T}$

However, [28] derived GRKM method for solving general quasi-linear ODEs of fifth-order. In general, the numerical methods can be preferred based on some of the following concepts: convergence, stability, and absolute or relative errors. In this paper, we have introduced one of these significant concepts: the stability of the numerical methods of RK type, which are described in Equations (4)-(11).

3. Main Types of Stability

Several notions of stability are used in numerical ODE analysis.

3.1 Zero Stability

It's important, particularly for multistep methods. A method is zero-stable if perturbations do not grow without bound when the differential equation is replaced by the corresponding difference equation.

For a linear multi step method (Lmm) with characteristic polynomial $\rho(\xi)=0$, the root condition is $|\xi| \leq 1$, and every root satisfying $|\xi| = 1$, [30]

3.2 Absolute Stability

Absolute stability studies the behavior of a numerical method when it is applied to a test equation.

For a fifth-order ODEs in Equation (1) the classical test equation is for a direct fifth-order RKM method in Equations (4)-(7), the natural test equation is instead

$$y^{(5)}(x) = \lambda y(x), \quad (22)$$

The method is absolutely stable for a particular value of the dimensionless parameter if all amplification factors satisfy $|\xi| \leq 1$. The corresponding set of stable values is called the region of absolute stability.[30].

3.3 A-Stability

A method is A-stable if its absolute-stability region contains the entire left half-plane:

$$\text{Re}(z) \leq 1,$$

Explicit Runge-Kutta methods cannot be A-stable in the classical first-order setting.

3.4 L-Stability

An A-stable method is L-stable if, in addition, $\lim_{|z| \rightarrow \infty} R(z) = 0$, where $R(z)$ is the stability function.

L-stability is particularly important for stiff ODEs.

3.5 Relative Stability

Relative stability concerns the numerical error compared with the exact solution, particularly for oscillatory problems.

3.6 Nonlinear B-Stability

For nonlinear problems, one may study whether perturbations are contractive under suitable monotonicity assumptions. This leads to concepts such as **B-stability**, contractivity, and algebraic stability [31-32].

4. Absolute Stability of Numerical Method of RKM type for Solving Odes of Fifth-Order

4.1 Fifth-Order Test Equation

For the direct fifth-order method, The test equation in Equation (22). It is often convenient to introduce $\lambda = \mu^5$ then, define $z = \mu h$. Consequently, $\lambda h^5 = z^5$.

For dissipative problems one may write $y^{(5)}(x) = -\mu^5 \lambda y(x)$, in which case $\lambda h^5 = -z^5$

4.2 Amplification Matrix

Introduce the scaled solution vector $U_n = \begin{pmatrix} y_n \\ h y_n' \\ h^2 y_n'' \\ h^3 y_n''' \\ h^4 y_n^{(4)} \end{pmatrix}$,

The exact Taylor-shift part of the method is represented by

$$P = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

From the test equation in (22), the RKM method produces $U_{n+1} = M(z) U_n$, where $M(z)$ is the amplification matrix.

The absolute-stability condition is therefore $\rho(M(z)) \leq 1$ where $\rho(M) = \max_j |\xi_j|$ is the spectral radius and ξ_j are the eigenvalues of M .

This is the key difference from the ordinary first-order RK method: for the direct fifth-order RKM method, stability is naturally a **matrix/eigenvalue problem**, rather than simply checking one scalar quantity $R(z) \leq 1$

4.3 Absolute Stability Polynomial

The amplification factors ξ are obtained from $|\xi I - M(z)| = 0$

Thus, for a direct fifth-order RKM method, the appropriate absolute stability polynomial is

$$P(\xi, z) = |\xi I - M(z)| = 0$$

Since M is a 5×5 matrix,

$$P(\xi, z) = \xi^5 + a_4(z) \xi^4 + a_3(z) \xi^3 + a_2(z) \xi^2 + a_1(z) \xi + a_0(z)$$

The five roots $\xi_1(z), \xi_2(z), \xi_3(z), \xi_4(z), \xi_5(z)$ are the amplification factors.

Therefore, RKM5 is absolutely stable at $z \Leftrightarrow |\xi_i(z)| \leq 1; \quad i=1,2,3,4,5$.

The fifth-order Taylor stability polynomial is

$$R_5(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!}$$

Its classical absolute-stability region is $S_{R_5(z)} = \{z \in \mathbb{C} : |R_5(z)| \leq 1\}$.

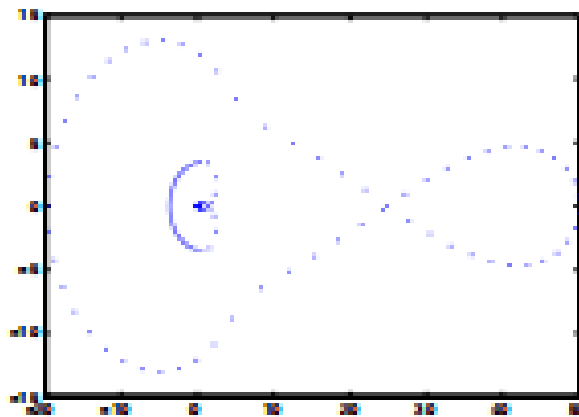


Figure 1: Stability Region of the Proposed Method

5. Conclusion

The study provides useful theoretical and computational insights into the stability properties of the numerical RKM method and its applicability to ODEs of fifth-order. Different types of stability of the numerical methods for solving ODEs have been introduced.

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