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A one sector model of economic growth with uncertain technology: An example of steady state analysis in a stochastic optimal control problem

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ABSTRACT

This paper considered the convergence of optimum levels of consumption and investment to a steady state to the case where output or technical progress is a random variable. The objective is the maximization of the expected value of the discounted sum of utilities facing uncertain technology or technological progress and, furthermore, our analysis is conducted in discrete time. Using the elementary mathematical technique, we established a stochastic analog of convergence to a steady state—the modified golden rule. The direction we took was to examine steady state or limiting behavior of the optimal control and state variables. The stochastic process (14) converges to 0 or ∞ in probability.

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1. Introduction

This work is an attempt to carry out the well-known results of Cass (2005), Koopmans (2005) concerning convergence of optimum levels of consumption and investment to a steady state to the case where output or technical progress is a random variable. We follow the above writers in concluding our analysis in a one-sector model. We departed from them by taking as our objective the maximization of the expected value of the discounted sum of utilities facing uncertain technology or technological progress and, furthermore, our analysis is conducted in discrete time.

Using elementary mathematical technique, we establish a stochastic analog of convergence to a steady state—the modified golden rule.

This paper is essentially a blend of dynamic programming and discrete time stochastic optimal control

theory. For those that are not interested in the economic issues involved, we feel that this work should be useful to illustrate some of the problems that arise in the analysis of economic models, in particular, models of economic growth.

Model of economic growth tends to possess a lot of nonlinear concavity in both the objective and constraint functions. The focus of the analysis may take several directions. One direction is to examine steady state or limiting behavior of the optimal control and state variables, which is the direction we shall take here.

2. Review of literature

The structure of optimal growth problems is related to that of various inventory models. Although the inventory theory does not seem to be directly applicable, learning the technique used is payoff enough for the

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growth theorist to spend some time on inventory theory. For example of a stochastic stability theorem in inventory theory, see [Karlin \(2008\)](#). However, Karlin's functional analytic techniques are beyond the reach of the average economist, our technique is well within the bound and furthermore, appeals to the intuition.

[Mirman \(2000\)](#) and [Mirrlees \(2005a; 2005b\)](#) are the works we have discovered that seem to be most relevant for our purpose. Mirrlees formulates a continuous time, one-sector neoclassical model where technical change is random, Harrod neutral, has exponentially growing mean, and its logarithm follows a Wiener process through time. His objective is maximization of the expectation of the discounted sum of utilities. In the main, he finds a differential equation to characterize optimal paths and proves existence of such. He does not investigate any kind of stochastic stability. Mirrlees operates in continuous time where the theory of stochastic processes is messy. We avoid the messier mathematics by working in discrete time.

[Mirman \(2000\)](#) investigated a class of stochastic difference equations that represent capital accumulation. These stochastic processes do not arise from any optimal growth model—at least not explicitly. Hence, it is important from an economic point of view to see if some kind of convergence to a steady state can be established for an optimal growth model.

3. Model

The model is set forth as follows:

$$\begin{aligned} & \text{Maximize } E \sum_{t=0}^{\infty} \delta^t u(t) \\ & \text{Subject to} \\ & c_0 + x_0 = 0 \\ & c_1 + x_1 = f(x_0, r) \\ & c_2 + 2 = f(x_1, r) \end{aligned} \quad (1)$$

Where $c_1, x_1, r, f(x, r), S, \delta, u, E$ are consumption at t , investment at t , positive valued random variable, production function, initial stocks, discount factor, utility function, and expected value operator, respectively. We assume u, f , strictly concave, differentiable and increasing, $u'(0) = \infty$, $f'(0, r) = \infty$, and $f'(\infty, r) < 1$ for all values of r , and $0 < \delta < 1$. For simplicity, we assume that $r: \Omega \rightarrow R$, where Ω is a finite set of elementary events and R is the real. The assumption that Ω is finite avoids messy mathematics. A generalization appears in [Brock and Mirman \(2001\)](#). Example follows.

For example, if investment is reversible, then

$$C_{t+1} + K_{t+1} - K_t = F(K_t, L^t, r) \quad (2)$$

May be put in the form [Eq. (1)] by

$$\frac{C_{t+1}}{L^{t+1}} + \frac{K_{t+1}}{L^{t+1}} = \frac{1}{L} \left[F\left(\frac{K_t}{L^t}, 1, r\right) + \frac{K_t}{L^t} \right]$$

If F is linear homogeneous in the first two variables, Define

$$c_t \frac{C_t}{L^t}, x_t \frac{K_t}{L^t}, f(x, r) = \frac{1}{L} F(x, 1, r) + 1$$

If u is homogeneous of some degree, and F satisfies the usual economic conditions, we are in the realm of Eq. (1). For another example, if technical change is random, Harrod neutral, i.e.,

$$F(K, L, t) = F(K, AL)$$

Where A is of the form $A = \lambda_r^t$, where r is random, $r: \Omega \rightarrow [a, b]$. If F is linear homogeneous and u is homogeneous of some degree, then a change of units puts us in the reach of Eq. (1).

Some caution is needed in formulating a model of random technical progress. That is in a reasonable world one expects that the worst in the future should beat the best today, i.e., in the random Harrod neutral case

$$\min_{\omega \in R} \lambda r(\omega) > \max_{\omega \in R} r(\omega) \quad (3)$$

Inequality [Eq. (3)] asserts that with probability 1, there will be technical progress—not regress and amounts to $\lambda > b/a$.

Solution of Eq. (1) produces, for each S , a sequence of optimal random variables $\bar{x}_0, \bar{x}_1, \bar{x}_2, \dots$ does this sequence converge in some sense? The most useful concept of convergence seems to be that of convergence in distribution. To explain, let $F_t(x) = \Pr\{\bar{x}_t < x\} \equiv$ probability that $\bar{x}_t < x$. F_t is called the distribution function of the random variable \bar{x}_t . The sequence of random variables $\bar{x}_0, \bar{x}_1, \bar{x}_2, \dots$ converges in distribution if for all $x \in R$, $F_t(x) \rightarrow F(x)$, $t \rightarrow \infty$. Here, R denotes the real numbers. Our main result is: Let F_t be the distribution function of \bar{x}_t , where $\{\bar{x}_t\}_{t=0}^{\infty}$ solves Eq. (1) then $F_t \rightarrow F$, $t \rightarrow \infty$ and F is independent of initial stock, S .

Much of the structure of our analysis can be displayed by first considering that simpler deterministic problem

$$\text{Maximize } \sum_{t=0}^{\infty} \delta^t u(c_t)$$

Subject to

$$\begin{aligned} c_0 + x_0 &= S \\ c_1 + x_1 &= f(x_0) \\ c_2 + 2 &= f(x_1) \end{aligned} \quad (4)$$

By strict concavity we have, for each S , exactly one $\bar{x}_0, \bar{x}_1, \dots$ that solves Eq. (4). Hence $x_0 = h(S)$. By straightforward dynamic programming considerations, e.g., see [Levhari and Srinivasan \(2009\)](#), we have the existence of a policy function h such that $\bar{x}_t = h[f(\bar{x}_{t-1})]$. Furthermore, following [Levhari and Srinivasan \(2009\)](#) it is straightforward to prove that h is increasing, continuous, $h(0) = 0$. Proving the convergence theorem $\bar{x}_t \rightarrow \bar{x}$, $t \rightarrow \infty$ is equivalent to proving that h has only one positive fixed point \bar{x} and that \bar{x} is stable. To dispose of this, let us examine the necessary condition of optimality

$$\begin{aligned} a(x_{t-1}) &\equiv u'[f(x_{t-1}) - x_t] = u'[f(x_{t-1}) - h(f(x_{t-1}))] \\ &= \delta u'[f(x_t) - h(f(x_t))] f'(x_t) \equiv b(x_t) \end{aligned} \quad (5)$$

Since optimal consumption $\bar{c}_{t+1} = f(\bar{x}_t) - h[f(\bar{x}_t)]$ increases in \bar{x}_t [dynamic programming as in [Levhari and Srinivasan \(2009\)](#), again], we see that $a(x)$, $b(x)$ decreases in x . Furthermore, $b(x)/a(x) = f'(x)$ decreases in x , $\frac{b(0)}{a(0)} = \infty$, $\frac{b(\infty)}{a(\infty)} = 0$. Hence, there is only one positive \bar{x} such that $a(\bar{x}) = b(\bar{x})$. If $a \equiv f^{-1}(S) < \bar{x}$, then \bar{x}_t increases to \bar{x} , if $a = \bar{x}$, then $\bar{x}_t = \bar{x}$, if $a > \bar{x}$, then \bar{x}_t decreases to \bar{x} .

The above analysis was terse and informal but its structural similarity should ease the reader's entry into the more difficult random case.

Analysis of the random case [Eq. (1)] proceeds as follows:

(a) We establish by dynamic programming the existence of a policy function h such that $\bar{x}_t = h[f(\bar{x}_{t+1}, r)] \equiv h_r(\bar{x}_{t+1})$, h is increasing and $h(0) = 0$.

(b) Let $r: \Omega \rightarrow R$ has minimum value 1 and maximum value n , where n is the number of points in Ω . Then, we study the structure of fixed points for the pair of functions $h_1(x)$, $h_n(x)$ by using the necessary conditions of optimality. Using the stochastic process $\bar{x}_t = h_r(\bar{x}_{t+1})$, we derive a relationship between

$$\begin{aligned} F_t(x) &\equiv \Pr\{\bar{x}_t < x\} \text{ and } F_{t-1}(x) \text{ by} \\ F_t(x) &= \sum_{S=1}^n \Pr[h_s(\bar{x}_{t-1}) < x] \alpha_s \\ &= \sum_{S=1}^n F_{t-1}[q_s(x)] \alpha_s \end{aligned} \quad (6)$$

Where q_s is the inverse function of h_s , and α_s = probability that r takes the value of S , $S = 1, 2, \dots, n$. Hence, any limit distribution function F must satisfy

$$F(x) = \sum_{S=1}^n F[q_s(x)] \alpha_s \quad (7)$$

Under conditions imposed by optimality, and general assumptions on $f(x, r)$, it will be shown that Eq. (7) has a unique solution and it is "stable." Unique solution of Eq. (7) amounts to convergence in distribution to the "stochastic modified golden rule." The details follow; first we put down all assumptions in one place.

Assumptions: $r: \Omega \rightarrow [1, 2, \dots, n]$ $\min_{\omega \in \Omega} r(\omega) = 1$, $\max_{\omega \in \Omega} r(\omega) = n$: $f(x, s)$ (a) increasing in s or (b) decreasing in s : to fix matters assume (a). $u', \frac{\delta f}{\delta x}(x, S) = f'(x, S)$ Both exist: u and f are strictly concave, $u'(0) = \infty$, $f'(0, S) = \infty$, $f'(\infty, S) < 1$ for $S = 1, 2, \dots, n$.

4. Lemmas and proofs

Lemma 1. Let $\bar{x}_0, \bar{x}_1, \bar{x}_2, \dots$ solve Eq. (1). Then, there is a policy function h such that $\bar{x}_t = h_r(\bar{x}_{t+1})$. Furthermore, h is increasing, $h(0) = 0$, $g[f(x, r)] = f(x, r) - h[f(x, r)]$ is continuous and increasing.

Proof: Follow [Levhari and Srinivasan \(2009\)](#). Continuity is straightforward.

Lemma 2. There is $\epsilon > 0$ such that on $(0, \epsilon)$, we have $h_1(x) > x$

Proof: Straightforward.

Lemma 3. If $x = \{\max\{x | h_s(x) = x\}\}$ exists then for all implies $h_s(x) < x$, i.e., the "last" fixed point of each h_s is stable.

Proof: straightforward.

This is another intuitively obvious lemma because if it were not true for some S_0 , then we can obtain unbounded capital accumulations from finite initial stock, $[a, b]$ is a stable interval if there are no fixed points of $h_1(x) > a$, no positive fixed points of $h_n(x)$ less than b , $h_1(a) = a$ and $h_n(b) = b$. It is obvious from the definition that if a stable interval exists, it is unique. Basically, a stable interval means that the Markov Process $x_t = h(x_{t-1}, r_t)$ is ergodic and acyclic. We give a heuristic definition of these terms. Ergodic means that the state space has only one interval $[a, b]$ possessing the property that once entered it is never left and no $[c, d] \subset [a, b]$ has that property. Acyclic means that there are no cyclically moving subclasses, i.e., between any two non-degenerate subintervals I_1, I_2 of $[a, b]$ there

is a positive probability path from I_1 to I_2 , in T steps is positive.

Lemma 4 is the most important lemma of this paper because we use information from the optimization process to demonstrate the existence of a stable interval. Recall that $h_1(x) < h_n(x)$ for each x , because $h_s(x)$ increases in S for each x .

Lemma 4. There is a stable interval. This lemma is the direct analog of the uniqueness of the fixed point in the deterministic case.

Proof: Let $g[f(x, r)] = f(x, r) - h[f(x, r)]$. By optimality of $\bar{x}_0, \bar{x}_1, \dots$ we have

$$u' \left[g(f(\bar{x}_{t-1}, r)) \right] = \delta \sum_{S=1}^n u' \left[g(f(\bar{x}_t, r)) \right] f'(\bar{x}_t, S) \alpha_S, \quad t = 1, 2, \dots \quad (8)$$

To understand Eq. (8) recall that $\bar{x}_t: \Omega \rightarrow [0, \infty)$; $r: \Omega \rightarrow R$, where Ω^t is the Cartesian product of Ω with itself t times. Now Eq. (8) says, in full: for each sample path $(\omega_1, \omega_2, \dots, \omega_t) \in \Omega^t$

$$\begin{aligned} & u' \left[g(f(\bar{x}_{t-1}, (\omega_1, \dots, \omega_{t-1})), r(\omega_t)) \right] \\ &= \delta \sum_{S=1}^n u' \left[g(f(\bar{x}_t(\omega_1, \dots, \omega_t) S)) \right] \\ & \quad f'(\bar{x}_t(\omega_1, \dots, \omega_{t-1}), S) \alpha_S, \quad t = 1, 2, \dots \end{aligned} \quad (9)$$

Needless to say, the compact form [Eq. (8)] is preferable to Eq. (9). Put $d[f(x, r)] \equiv u'[g(f(x, r))]$. Consider the fixed points $X_i = h[f(S_i, i)]$ $i = 1, 2, \dots, n$. Since the stochastic process described by the necessary condition of optimality [Eq. (8)] is the same as $\bar{x}_t = h[f(\bar{x}_{t-1}, r)]$ it is straightforward to prove for any fixed point $X_i = h[f(X_i, i)]$ that

$$d[f(x_i, i)] = \delta \sum_{S=1}^n d[f(x_i, S)] f'(x_i, S) \alpha_S \quad (10)$$

$$i = 1, 2, \dots, n$$

Now d is decreasing hence, $f(x_i, 1) < f(x_i, S) < f(x_i, n)$ implies $d[f(x_i, 1)] > d[f(x_i, S)] > d[f(x_i, n)]$, $S = 2, 3, \dots, n-1$. Therefore, from Eq. (10)

$$\begin{aligned} & d[f(x_1, 1)] < \delta \sum_{S=1}^n d[f(x_1, 1)] f'(x_1, S) \alpha_S \text{ and} \\ & d[f(x_n, n)] > \delta \sum_{S=1}^n d[f(x_i, S)] f'(x_i, S) \alpha_S \end{aligned}$$

Thus

$$1 < \delta \sum_{S=1}^n f'(x_1, S) \alpha_S$$

and (11)

$$1 > \delta \sum_{S=1}^n f'(x_n, S) \alpha_S$$

Suppose now that there exists another pair of fixed points x'_1, x'_n such that $x_1 < x_n \leq x'_1 < x'_n$. Now x'_1, x'_n must satisfy Eq. (11) because $f(x, S)$ decreases in X ,

$$1 < \delta \sum_{S=1}^n f'(x'_1, S) \alpha_S \leq \delta \sum_{S=1}^n f'(x_n, S) \alpha_S < 1 \text{ a contradiction. The lemma is established.}$$

Remark: The proof for Lemma 4 for this case where $f(x, S)$ is decreasing in S is similar. Lemma 4 assures us that x_1, x_n is a pair of fixed points such that there are no fixed points of h_n to the right of x_n and on fixed points of h_n to the left of x_1 .

Intuition plus a little reflection makes it plausible that the stochastic process $\bar{x}_t = h_r(\bar{x}_{t-1})$ will settle in the range $[x_1, x_n]$ as $t \rightarrow \infty$. Notice also that the range $[x_1 < x_n]$ is independent of initial stock S .

Theorem: Let $F_t(x) = \Pr[\bar{x}_t < x]$. Then, there is a distribution function F such that for all $x \in R$, $F_t(x) = F(x)$, $t \rightarrow \infty$. Furthermore, F does not depend on S .

Proof: Let \bar{x}_n be optimal initial investment. It is a straightforward job to prove that for any $\bar{x}_0 > 0$ the sequence

$$F_t(x) = \sum_{S=1}^n F_{t-1}[q_S(x)] \alpha_S, \quad t = 1, 2, \dots \quad (12)$$

Converges uniformly on the line R . Here, $F_0(x)$ is the degenerate distribution corresponding to \bar{x}_0 , i.e., $F_0(x) = 0$ for $x \leq \bar{x}_0$ and $F_0(x) = 1$ for $x > \bar{x}_0$. The prove uniformly converges and we show that the sequence $F_{t,t=0}$ defined by Eq. (12) is uniform Cauchy, i.e., given $\varepsilon > 0$ there is N_ε we have $\sup_{x \in R} \{ |F_{p+k}(x) - F_p(x)| \} < \varepsilon$.

To do this notice that by iterating Eq. (12) we have

$$\begin{aligned} F_{p+k}(x) - F_p(x) = & \sum_{i_{p+k}, \dots, i_{p+1}} \sum_{i_p, \dots, i_1} \left\{ F_0 \left[q_{i_{p+k}} q_{i_{p+k-1}} \dots q_{i_p} \dots q_{i_1}(x) \right. \right. \\ & \left. \left. - F_0 \left[q_{i_p} \dots q_{i_1}(x) \right] \right\} \alpha_{i_p} \dots \alpha_{i_1} \alpha_{i_{p+k}} \dots \alpha_{i_{p+1}} \end{aligned} \quad (13)$$

Where each i_i may take the value $1, 2, \dots, n$. Hence, there are n^{p+k} items summed in the inner brackets. Notice that entities such as $q_{i_p} \dots q_{i_1}(x)$ are simple realizations of the stochastic process

$$y_t = q(y_{t-1}, r) = q_r(y_{t-1}), y_0 = x \quad (14)$$

Which is the “inverse” process of $\bar{x}_t = h[f(x_{t-1}, r)]$. The stochastic process [Eq. (14)] converges to 0 or ∞ in probability. To see this if we start with x then one period later $q_{i_1}(x)$ emerges with probability α_{i_1} , two periods later $q_{i_2}q_{i_1}(x)$ emerges with probability $\alpha_{i_2}\alpha_{i_1}$, etc. Notice that $[x_{1m}, x_{nm}]$ is transient, i.e., once it is left it is never returned to and it is left with positive probability. Hence, after a long period of time “most” realizations of Eq. (14) leave $[x_{1m}, x_{nm}]$. Now if the interval $[0, x_{1m}]$ is entered by a realization of Eq. (14), i.e., there is p, i_1, \dots, i_p such that $x_p = q_{i_p}, \dots, q_{i_1}(x) \in [0, x_{1m}]$, then $x_{p+1} = \max_{1 \leq i \leq n} q_i(x_p) < x_p$ and the sequence $x_{p+j} = \max_{1 \leq i \leq n} q_i(x_{p+j-1})$ converges monotonely to 0 as $j \rightarrow \infty$. i.e., the process [Eq. (14)] is monotone decreasing to 0 on $[0, x_{1m}]$. Similarly, the process [Eq. (14)] is monotone increasing to ∞ on (x_{nm}, ∞) .

Let us now return to Eq. (13). If p is large “most” of the $q_{i_p}, q_{i_{p-1}}, \dots, q_{i_1}(x)$ are in $[0, x_{1m}]$ or (x_{nm}, ∞) and if $q_{i_p}, \dots, q_{i_1}(x)$ is in $[0, x_{1m}]$ or (x_{nm}, ∞) then so is $q_{i_{p+k}}, \dots, q_{i_1}(x)$ in $[0, x_{1m}]$ or (x_{nm}, ∞) , respectively. Thus, if $\bar{x}_0 > 0$ and since $F_0(x) = 0, x \leq \bar{x}_0$ and $F_0(x) = 1, x > \bar{x}_0$ then there will be a p_n large enough so that for $p \leq p_n, k = 0, 1, 2, \dots$ “Most” of the $0, x_{1m}$ $F_0[q_{i_p}, \dots, q_{i_1}(x)] = F_0[q_{i_{p+k}}, \dots, q_{i_1}(x)]$. Hence, the difference in Eq. (13) goes to 0 as $p \rightarrow \infty$. Furthermore, a little reflection will convince one that convergence is uniform.

Hence, there is a limit, F , such that $F_i = F$ uniformly. Thus, F is non-decreasing. It is trivial to prove, by exploitation of Eq. (12) that $F = 1$ and $(x_{nm}, \infty) F = 0$ on $[0, x_{1m}]$. Also, F is continuous from the left as it is a uniform limit of continuous from the left F_t . Hence, F is a distribution. F must satisfy

$$F(x) = \sum_{s=1}^n F[q_s(x)] \alpha_s \quad (15)$$

To see that the limit F is independent of initial stocks S , we need only show that there is only one distribution F that satisfies Eq. (15). The Eq. (15) has a unique solution. For if F and G are two distributions that solve Eq. (15) then

$$\begin{aligned} H(x) &= F(x) - G(x) = \sum_{i_1} H[q_{i_1}(x)] \alpha_{i_1} \\ &= \sum_{i_1, i_2} H[q_{i_2}, q_{i_1}(x)] \end{aligned} \quad (16)$$

$$\alpha_{i_2} \alpha_{i_1} = \sum_{i_p, \dots, i_1} H[q_{i_p}, q_{i_1}(x)] \alpha_{i_p}, \dots, \alpha_{i_1}$$

Now, if F, G solve Eq. (15), it is trivial to prove that $F = G = 0$ on $[0, x_{1m}]$ and $F = G = 1$ on (x_{nm}, ∞) . Hence, an argument exactly like the one given for the convergence of the F_t given by Eq. (12) shows that the right hand side of Eq. (10) goes to zero uniformly in X as $p \rightarrow \infty$.

5. Remarks

Mirman (2001) has extended his work in Mirman (2000) to include the case of processes with a stable interval. Thus, his class of stochastic processes in Mirman (2001) includes process arising from optional growth models of the type studied in this paper. A generalization of the results of this paper to general random variables is in Brock and Mirman (2001). The case $\delta = 1$ is treated in Brock and Mirman (2001) for a one good model like the one in this paper. Jeanjean (2001) has extended some of these results to multi-sector models.

In this paper, we chose to take the self-contained route in order to delineate the ideas and to reach a wider audience. However, one may apply tools from probability theory to prove ergodicity and acyclicity, see Mirman (2001) for this and a list of references to the relevant Markov theoretic articles.

6. Recommendation

1. Let us expand upon this point. If capital cannot be eaten into after it is installed, then the constraint $K_{t+1} \geq K_t, t = 0, 1, \dots$ must be added to the optimization problem. In this case, it does not follow that the problem may be put into form [Eq. (1)] and it will give the same optimal solution as Eq. (1) does. By assuming that investment is reversible, we allow the possibility that $K_{t+1} \geq K_t$ for some time t . We conjecture that our results can be proved in essentially the same form for the case where the constraint $K_{t+1} \geq K_t$ is present.
2. Notice that Eq. (11) is the stochastic analog to the condition $1 = \delta f'(\bar{x})$ which is the equation determining the modified golden rule in the deterministic case.

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