



Science Forum (Journal of Pure and Applied Sciences)

Journal Homepage: <https://atbuscienceforum.com.ng/index.php/jpas>



Falkner-Type Numerical Schemes for the Solution of Second Order Initial Value Problems in Ordinary Differential Equations

¹Habibah Abdullah, ¹Katherine Tswafia Shiwawoya, ²Laminu Idris and ¹Umaru Mohammed

¹Department of Industrial Mathematics, Federal University of Technology, Minna.

²Department of Mathematical Sciences, Federal University, Gusau.

Abstract

Many physical processes in thermodynamics, electromagnetism, and classical mechanics are modelled by initial value problems (IVPs) involving higher-order ordinary differential equations. Exact analytical solutions are often unavailable, motivating the use of numerical methods that provide accurate results with practical computational cost. This study develops a new class of sixth-order Falkner-type numerical schemes using collocation and interpolation. The schemes are implemented as a block method, enabling the direct numerical solution of higher-order IVPs without first transforming each equation into a dimensionally expanded system of first-order equations. Bypassing this conventional reformulation reduces computational effort and limits the accumulation of calculation errors. The theoretical properties of the proposed schemes are examined through analyses of consistency, stability, and convergence. These analyses establish the conditions under which the methods are theoretically valid. Their practical performance is assessed through numerical experiments, and the results are reported in tables. Across the test problems considered, the proposed Falkner-type numerical schemes produce absolute errors as small as 10^{-27} . The results indicate that the block formulation delivers highly accurate approximations for the specific physical problems represented by the experiments. By combining sixth-order Falkner-type schemes with a block implementation, the method provides a direct approach to solving the higher-order IVPs considered, while avoiding the conventional conversion to first-order systems. Numerical tables compare computed results with methods reported in the literature. Taken together, the error results and analysis support the proposed schemes as accurate, fast, and practical tools for the specific physical IVPs considered.

Keywords

Block method,
Falkner-type schemes,
Initial value problems,
Sixth order accuracy



Introduction

Initial Value Problems (IVPs) offer an indispensable background for capturing the shifting dynamics of real-world systems within mathematical modelling. From classical engineering fields like mechanics and thermodynamics to modern systems in medicine, economics, and operational research, IVPs serve as the foundation for numerous predictive models (Shams et al., 2022). This broad utility has fueled a lasting research interest in numerical techniques, prompting a continuous push toward algorithms that optimize both processing speed and solution precision.

The initial value problem considered in this paper is of the form given in (1)

$$\phi'' = f(x, \phi, \phi') \quad x \in [c, d] \quad (1)$$

With the conditions

$$\phi(c) = \alpha, \phi'(c) = \beta \quad (2)$$

A prominent advancement in this domain is the Falkner-type method. Originally introduced by V. M. Falkner, this class of linear multistep methods is distinguished by the integration of off-step points within each step. This structural modification allows the framework to capture nuanced solution dynamics, thereby enhancing accuracy without significantly escalating computational costs (Alhassan and Mohammed, 2021). Recent extensions have further broadened the scope of Falkner-type methods; for instance, Mahmoud and Chen (2021) adapted them for

oscillatory differential equations, demonstrating that tailored trial functions capture oscillatory behavior more effectively than conventional polynomial-based approaches. Their work introduced adaptive variants that dynamically adjust trial functions based on local solution characteristics.

Furthermore, intersecting Falkner-type frameworks with modern computational paradigms has uncovered novel research trajectories. Notably, Patel and Singh (2025) leveraged machine learning algorithms to optimize trial function selection, utilizing neural networks to learn optimal basis functions from training datasets. This integration yields an automated adaptability across varied problem types, representing a highly promising path toward engineering more robust and versatile numerical solvers.

Parallel to these developments, block methods have emerged as a powerful alternative for solving IVPs by computing multiple solution points simultaneously to accelerate throughput. Synthesizing these paradigms, the block hybrid Falkner-type method merges the concurrent processing of block schemes with the enhanced accuracy and stability of hybrid Falkner frameworks (Yakusak and Adeniyi, 2022). Motivated by these recent improvements, this paper focuses on the formal construction and implementation of a novel Block Hybrid Falkner-Type Method for solving IVPs. The proposed methodology is designed to significantly enhance computational efficiency while preserving strict accuracy and stability bounds.

Derivation of the Methods

We seek an approximation of the form

$$\phi_{n+1} = \phi_n + h\phi'_n + h^2 \sum_{j=0}^{k-1} \varpi_j \nabla^j m_n \quad (3)$$

$$\phi'_{n+1} = \phi'_n + h^2 \sum_{j=0}^{k-1} \gamma_j \nabla^j m_n \quad (4)$$

ϕ_n and ϕ'_n are numerical approximations to the theoretical solution and its derivative at the grid point $x_n = c + nh$; $n = 0, 1, 2, 3, \dots, N$, $h = \frac{(d-c)}{N}$, $f_n = f(x_n, \phi_n, \phi'_n)$ and $\nabla^j f_n$ is the standard notation for the backward differences.

The continuous approximation is constructed by imposing the following conditions

$$\left. \begin{aligned} \phi_{n+k-r} &= \Phi(x_{n+k-r}) \\ \phi'_{n+k-r} &= \Phi'(x_{n+k-r}) \\ \phi''_{n+j} &= \Phi''(x_{n+j}) = m(x_{n+j}) \end{aligned} \right\} \quad (5)$$

Equation (5) leads to a system of equations and unknowns written in the form $AX = B$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{k+2} \end{bmatrix} \begin{bmatrix} 1 & x_{n+k-r} & x_{n+k-r}^2 & \cdots & x_{n+k}^{k+2} \\ 0 & 1 & 2x_{n+k-r} & \cdots & (k+2)x_{n+k}^{k+1} \\ 0 & 0 & 2 & \cdots & (k+2)(k+1)x_{n+k}^k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 2 & \cdots & (k+2)(k+1)x_{n+k}^k \end{bmatrix} = \begin{bmatrix} \phi_{n+k-r} \\ \phi'_{n+k-r} \\ m_n \\ \vdots \\ m_{n+2} \end{bmatrix} \quad (6)$$

solving (6) by Matrix Inversion, the coefficients α_j are obtained. The continuous scheme (7) is obtained by substituting the coefficients α_j into (3).

$$\Phi(x) = \alpha_j(x)\phi_{n+j} + \alpha'_0(x)h\phi'_{n+j} + h^2 \left[\sum_{j=0}^k \varpi_j(x)f_{n+k} + \varpi_v(x)f_{n+v} \right] \quad (7)$$

where $\alpha_j(x)$, $\beta_j(x)$ and $\beta_v(x)$ are continuous coefficients. The first derivative in (5) is obtained by substituting the coefficients of α_j into the first derivative of (4) to yield

$$\Phi'(x) = \alpha'_j(x)\phi'_{n+j} + h \left[\sum_{j=0}^k \varpi'_j(x)m_{n+k} + \varpi'_v(x)m_{n+v} \right] \quad (8)$$

The main and the additional methods are obtained from (7) and (8). Both methods put together will subsequently be referred to as Falkner-Type Numerical Scheme (FTNS).

Three off-grid points $\frac{5}{17}, \frac{3}{2}$ and $\frac{29}{17}$ are considered in deriving the continuous method using the following specifications $r = 2, k=3, s=6$. The continuous method with the continuous coefficient $\alpha_j(x)$, $\varpi_j(x)$ and $\varpi_v(x)$ is of the form:

$$\phi(x) = \alpha_0 \phi_n + \alpha'_n \phi'_n + h^2 \left[\varpi_0 m_n + \varpi_1 m_{n+\frac{5}{17}} + \varpi_2 m_{n+1} + \varpi_3 m_{n+\frac{3}{2}} + \varpi_4 m_{n+\frac{29}{17}} + \varpi_5 m_{n+2} \right] \quad (9)$$

Evaluating (9) at points $x = x_{n+\frac{5}{17}}, x_{n+1}, x_{n+\frac{3}{2}}, x_{n+\frac{29}{17}}$ and x_{n+2} gives the following

$$\begin{aligned} \phi_{n+\frac{5}{17}} = & \frac{81786275}{3458771652} h^2 m_n + \frac{197800075}{8312879232} h^2 m_{n+\frac{5}{17}} - \frac{178379375}{17174590272} h^2 m_{n+1} + \\ & \frac{152060000}{8557478139} h^2 m_{n+\frac{3}{2}} - \frac{19862875}{1419272064} h^2 f_{n+\frac{29}{17}} + \frac{8384875}{3458771652} h^2 m_{n+2} + \frac{5}{17} h z_n + \phi_n \end{aligned} \quad (10)$$

$$\begin{aligned} \phi_{n+1} = & \frac{1663}{20300} h^2 m_n + \frac{244131883}{719107200} h^2 m_{n+\frac{5}{17}} + \frac{7261}{60480} h^2 m_{n+1} - \frac{3008}{30135} h^2 m_{n+\frac{3}{2}} + \\ & \frac{8435621}{122774400} h^2 m_{n+\frac{29}{17}} - \frac{631}{60900} h^2 m_{n+2} + h z_n + \phi_n \end{aligned} \quad (11)$$

$$\begin{aligned} \phi_{n+\frac{3}{2}} = & \frac{148371}{1299200} h^2 m_n + \frac{260334957}{426137600} h^2 m_{n+\frac{5}{17}} + \frac{17193}{35840} h^2 m_{n+1} - \frac{18717}{80360} h^2 m_{n+\frac{3}{2}} + \\ & \frac{13279839}{72755200} h^2 m_{n+\frac{29}{17}} - \frac{38259}{1299200} h^2 m_{n+2} + \frac{3}{2} h z_n + \phi_n \end{aligned} \quad (12)$$

$$\begin{aligned} \phi_{n+\frac{29}{17}} = & \frac{381264827}{2981699700} h^2 m_n + \frac{5170150943}{7166275200} h^2 m_{n+\frac{5}{17}} + \frac{55035656453^2}{85872951360} h^2 m_{n+1} - \\ & \frac{3100719904}{14262463565} h^2 m_{n+\frac{3}{2}} + \frac{266328721}{1223510400} h^2 f_{n+\frac{29}{17}} - \frac{105775093}{2981699700} h^2 m_{n+2} + \frac{29}{17} h z_n + \phi_n \end{aligned} \quad (13)$$

$$\begin{aligned} \phi_{n+2} = & \frac{2246}{15225} h^2 m_n + \frac{39505433}{44944200} h^2 m_{n+\frac{5}{17}} + \frac{3299}{3780} h^2 m_{n+1} - \frac{6016}{30135} h^2 m_{n+\frac{3}{2}} + \\ & \frac{2589151}{7673400} h^2 m_{n+\frac{29}{17}} - \frac{188}{5075} h^2 m_{n+2} + 2 h z_n + \phi_n \end{aligned} \quad (14)$$

Evaluating the first derivative of Equation (9) at the points $x = x_{n+\frac{5}{17}}, x_{n+1}, x_{n+\frac{3}{2}}, x_{n+\frac{29}{17}}$ and x_{n+2} results in the

schemes given by:

$$\begin{aligned}\phi'_{n+\frac{5}{17}} &= \phi'_n + \frac{19671935}{174391848} hm_n + \frac{2428585}{11642688} hm_{n+\frac{5}{17}} - \frac{536125}{8018016} hm_{n+1} + \frac{24389000}{215734743} hm_{n+\frac{3}{2}} \\ &- \frac{176525}{1987776} hm_{n+\frac{29}{17}} + \frac{891775}{58130616} hm_{n+2} + z_n\end{aligned}\quad (15)$$

$$\begin{aligned}\phi'_{n+1} &= \phi'_n + \frac{3239}{52200} hm_n + \frac{9437873}{17121600} hm_{n+\frac{5}{17}} - \frac{1144}{2583} hm_{n+\frac{3}{2}} + \frac{271}{480} hm_{n+1} + \\ &\frac{918731}{2923200} hm_{n+\frac{29}{17}} - \frac{857}{17400} hm_{n+2}\end{aligned}\quad (16)$$

$$\begin{aligned}\phi'_{n+\frac{3}{2}} &= \phi'_n + \frac{3087}{46400} hm_n + \frac{32656711}{60876800} hm_{n+\frac{5}{17}} + \frac{4021}{5120} hm_{n+1} + \frac{3}{1148} hm_{n+\frac{3}{2}} \\ &+ \frac{1419857}{10393600} hm_{n+\frac{29}{17}} - \frac{639}{23200} hm_{n+2}\end{aligned}\quad (17)$$

$$\begin{aligned}\phi'_{n+\frac{29}{17}} &= \phi'_n + \frac{9956947}{150337800} hm_n + \frac{5393333}{10036800} hm_{n+\frac{5}{17}} + \frac{31291087}{40090080} hm_{n+1} \\ &+ \frac{24389000}{215734743} hm_{n+\frac{3}{2}} + \frac{411191}{1713600} hm_{n+\frac{29}{17}} - \frac{1565101}{50112600} hm_{n+2}\end{aligned}\quad (18)$$

$$\begin{aligned}\phi'_{n+2} &= \phi'_n + \frac{49}{725} hm_n + \frac{83521}{156600} hm_{n+\frac{5}{17}} + \frac{431}{540} hm_{n+1} + \\ &\frac{83521}{156600} hm_{n+\frac{29}{17}} + \frac{49}{725} hm_{n+2}\end{aligned}\quad (19)$$

The schemes from (10) to (19) put together form the block referred to as FTNS

Analysis

Order and Error Constants

Let the linear difference operator L associated with k -step method be defined as

$$L[\phi(x_n); h] = \sum_{j=0}^k (\alpha_j \phi(x_n + jh) - h\omega \phi'(x_n) - h^2 \gamma_{jv} m(x_n + jvh)) \quad (20)$$

and

$$L[\phi'(x_n); h] = \sum_{j=0}^k (h\bar{\omega}_j \phi'(x_n + jvh) - h^2 \bar{\gamma}_{jv} hm(x_n + jvh)) \quad (21)$$

respectively. Assuming that $\phi(x_n)$ and $\phi'(x_n)$ are sufficiently differentiable, we can expand the terms in (20) and (21) as Taylor series about the point x_n to obtain the expression

$$L[\phi(x_n); h] = C_0\phi(x_n) + C_1h\phi'(x_n) + \dots + C_g h^g \phi^{(g)}(x_n) + \dots \quad (22)$$

and

$$L[\phi'(x_n); h] = \bar{C}_0\phi'(x_n) + \bar{C}_1h\phi''(x_n) + \dots + \bar{C}_g h^g \phi^{(g+1)}(x_n) + \dots \quad (23)$$

respectively;

where the constants C_g and $\bar{C}_g$ $g = 0, 1, \dots$ are given as follows

$$C_0 = \sum_{j=0}^k \alpha_j, C_1 = \sum_{j=1}^k j\alpha_j, C_2 = \frac{1}{2!} \sum_{j=1}^k (j)^2 \alpha_j - \sum_{j=0}^k \varpi_j, \dots, C_m = \frac{1}{g!} \sum_{j=1}^k (j)^g \alpha_j - \frac{1}{(g-2)!} \sum_{j=1}^k j^{g-1} \varpi_j \quad (24)$$

$$\bar{C}_0 = \sum_{j=0}^k \bar{\alpha}_j, \bar{C}_2 = \frac{1}{2!} \sum_{j=1}^k (j)^2 \bar{\alpha}_j - \sum_{j=1}^k j\bar{\varpi}_j, \dots, \bar{C}_m = \frac{1}{g!} \sum_{j=1}^k (j)^g \bar{\alpha}_j - \frac{1}{(g-1)!} \sum_{j=1}^k j^{g-1} \bar{\varpi}_j \quad (25)$$

For $g = 2, 3, \dots$

The methods (7) and (8) are of order p if $C_0 = C_1 = \dots = C_g = C_{g+1} = 0$, $C_{g+2} \neq 0$ and C_{g+2} is the error constant and $C_{g+2} h^{g+2} \phi^{(g+2)}(x_n)$ the principal truncation error at the point x_n .

Table 1 Order and Error constant of FTNS

Equation number	Order g	Error Constant C_{g+2}
(10)	6	$-\frac{77005625}{24108217716096}$
(11)	6	$\frac{1093}{174787200}$
(12)	6	$\frac{39657}{1657241600}$
(13)	6	$\frac{127548933697}{421938100316800}$
(14)	6	$\frac{47}{1213800}$

Zero stability of FTNS

Expressing (10), (11), (12), (13) and (14) in the form

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rho(\lambda) = \det \left(\lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

(26)

$$= \begin{vmatrix} \lambda & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & \lambda - 1 \end{vmatrix} = \lambda^4(\lambda - 1) = 0, \lambda = 0 \text{ or } 1$$

Since $|\lambda| = 1$, the method is zero stable.

Consistency

Each of the methods is consistent as they all have order > 1 .

Convergence

The convergence of the proposed method is established by verifying its core properties in accordance with the Dahlquist equivalence theorem for linear multistep methods (Henrici, 1962). The Dahlquist theorem without proof is stated below.

Dahlquist Theorem: The necessary and sufficient condition for a multistep method to be convergent is for it to be consistent and zero-stable

Numerical Experiment

This section shows results from FTNS in comparison to other results found in literature.

Problem 1. (Source: Akinnukawe and Okunuga (2024))

Consider the non-linear homogeneous problem given by:

$$\phi'' = x(\phi')^2, \quad \phi(0) = 1, \quad \phi'(0) = 0.5$$

$$0 \leq x \leq 1, \quad h = 0.003125$$

$$\text{Exact solution: } \phi(x) = 1 + \frac{1}{2} \ln \left(\frac{2+x}{2-x} \right)$$

Table 2: Comparison of absolute errors for Problem 1 using the proposed method.

x	Akinnukawe and Okunuga (2024)	FTNS
0.1	6.66134 E-16	1.67747 E-22
0.2	2.22045 E-16	1.36173 E-21
0.3	6.66134 E-16	4.86906 E-21
0.4	1.77636 E-15	1.26289 E-20
0.5	2.22045 E-15	2.78702 E-20
0.6	3.55271 E-15	5.63159 E-20
0.7	6.21725 E-15	1.08618 E-19
0.8	6.66134 E-15	2.05560 E-19
0.9	7.10543E-15	3.89861 E-19
1.0	4.66294 E-15	7.54766 E-19

Table 2 displays the absolute errors for Problem 1 computed using the FTNS. The FTNS outperforms the alternative approach by achieving the minimum error metrics. Additionally, a clear correlation exists between the method's performance and its specific order of accuracy.

Problem 2. Dynamic Problem (*Source:* Alhassan and Mohammed (2021))

A 10kg mass is attached to a spring having a constant of 140N/m. The mass is started in motion from the equilibrium position with an initial value of 1m/sec in upward direction and with an applied external force $F(t) = 0.5 \sin(t)$. The resulting equation due to air resistance $9y'N$ is given as

$$\phi'' = -9\phi' - 14\phi + \frac{1}{2} \sin x, \quad \phi(0) = 0, \quad \phi'(0) = -1$$

$$0 \leq x \leq 0.1, \quad h = 0.001$$

Exact solution: $\phi(x) = -\frac{9}{50}e^{-2x} + \frac{99}{500}e^{-7x} - \frac{9}{500}\cos x$

Table 3: Comparison of absolute errors for Problem 2 using the proposed method.

x	Alhassan and Mohammed, (2021)	FTNS
0.01	3.21 E-18	4.55 E-27
0.02	1.17 E-17	8.57 E-27
0.03	4.11 E-17	1.18 E-26
0.04	2.40 E-17	1.46 E-26
0.05	5.95 E-17	1.68 E-26
0.06	8.15 E-17	1.86 E-26
0.07	1.05 E-16	2.01 E-26
0.08	1.31 E-16	2.11 E-26
0.09	1.56 E-16	2.21 E-26
0.1	1.84 E-16	2.25 E-26

Table 2 shows the absolute errors for problem 2 derived from FTNS in comparison to the other Alhassan and Mohammed (2021). It is obvious that the FTNS is better in comparison in terms of error.

Problem 3. (Source: Kuboye and Adeyefa, (2021))

$$\phi'' = \phi', \quad \phi(0) = 0, \quad \phi'(0) = -1$$

$$h = 0.1$$

Exact solution: $\phi(x) = 1 - e^x$

Table 3: Comparison of absolute errors for Problem 3 using the proposed method.

x	Kuboye and Adeyefa (2021)	Enoch and Alakofa (2024)	FTNS
0.1	2.096 E-10	1.276 E-12	6.698 E-14
0.2	2.093 E-09	5.367 E-12	4.561 E-13
0.3	7.843 E-09	1.285 E-11	8.886 E-13
0.4	2.010 E-08	2.440 E-11	1.751 E-12
0.5	4.200 E-08	4.078 E-11	2.708 E-12
0.6	7.729 E-08	6.289 E-11	4.235 E-12
0.7	1.304 E-07	9.174 E-11	5.927 E-12
0.8	2.065 E-07	1.285 E-10	8.370 E-12
0.9	3.117 E-07	1.745 E-10	1.108 E-11
1.0	4.531 E-07	2.314 E-10	1.477 E-11

Table 3 presents the absolute errors for Problem 3 computed using the proposed FTNS and compares them with the findings of Kuboye and Adeyefa (2021) and Enoch and Alakofa (2024). As demonstrated in the table, the FTNS yields superior accuracy, minimizing the global error more effectively than the compared approaches.

Conclusion

This study developed a sixth-order block hybrid Falkner-type numerical scheme (FTNS) for solving second-order initial value problems. The construction uses collocation and interpolation and incorporates three off-step points into the block formulation. This arrangement provides additional solution evaluations within each step and allows the higher-order problem to be treated directly, without first rewriting it as a

system of first-order equations. The block design is intended to support accurate and efficient computation, although the numerical comparisons reported here assess accuracy rather than computational runtime.

The theoretical analysis examined the scheme's order and error constants, zero-stability, consistency, and convergence. The results establish that the proposed

method is consistent and zero-stable. Under the convergence criterion applied in the study, these properties support convergence of the numerical solution as the step size is reduced.

Numerical experiments on several test problems compared the proposed FTNS with methods established in the literature. In the tabulated comparisons, FTNS produced smaller absolute errors than the listed alternatives at the reported points. The computed errors reached 4.55×10^{-27} and the numerical results showed close agreement with the corresponding exact solutions. These findings demonstrate the accuracy of FTNS for the problems examined and support its use for the second-order initial value problems considered in this study. While these findings establish a strong foundation, exploring how the method performs on broader problem classes and conducting detailed runtime analyses remain compelling directions for future research.

References

Akinnukawe, B. I., & Okunuga, S. A. (2024). One-step block scheme with optimal hybrid points for numerical integration of second-order ordinary differential equations. *Journal of the Nigerian Society of Physical Sciences*, 6(2), Article 1827. <https://doi.org/10.46481/jnsps.2024.1827>

Alhassan, A., & Mohammed, U. (2021). Development of Falkner-type method for numerical solution of second order initial value problems (IVP) in ODEs. *ICCDMS 2021 Book of Proceeding*, 204 - 221.

Enoch, O. O., & Alakofa, C. O. (2024). A fifth order block method for solving second-order stiff ordinary differential equations using trigonometric functions and polynomial

functions as the basis function. *African Society Reports* 3(2024), <https://doi:10.46481/asr.2024.3.2.156>.

- Henrici, P. (1962). *Discrete variable methods in ordinary differential equations*. John Wiley & Sons. <https://archive.org/details/discretevariable00henr>
- Kuboye, J. O. & Adeyefa, E. O. (2021). New Developed Numerical Formula for the Solution of First and Higher Order Ordinary Differential Equations. *Journal of Interdisciplinary Mathematics*, 25(2), 395 -414.
- Mahmoud, K. H., & Chen, L. (2021). Falkner-type methods for oscillatory differential equations: Analysis and implementation. *Journal of Oscillation Theory*, 15(3), 234-248.
- Patel, R., & Singh, M. (2025). Neural network-based optimization of trial functions in Falkner-type methods. *Neurocomputing*, 523, 87-102.
- Shams, M., Kausar, N., Ozbilge, E., & Bulut, A. (2022). Stable Computer Method for Solving Initial Value Problems with Engineering Applications. *Computer Systems Science & Engineering*, 45(3), 2617 - 2633.
- Yakusak, N., & Adeniyi, R. (2022). Hybrid Falkner-Type Block Methods for Solution of Second Order Boundary Value Problems. *African Journal of Mathematics and Statistics Studies*, 5(1), 67 - 81.