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Development of a Two-Step Block Numerov Method with Bhaskara Hybrid Points for Second Order Initial Value Problems

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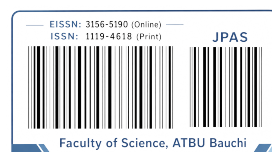
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Abstract

This study presents a two-step block Numerov method for the direct numerical solution of second-order initial value problems. The formulation incorporates two Bhaskara hybrid off-grid points to improve the representation of the solution within each integration step. A power-series polynomial is constructed and interpolated, then collocated at selected grid, off-grid, and hybrid points. The resulting continuous scheme is used to derive a block of discrete schemes, which are solved simultaneously to advance the numerical solution. The method produces a block whose component formulas have orders five, five, and six, respectively. Its principal analytical properties are examined, including the error constants associated with the formulas, consistency, zero stability, and convergence. These results establish the method's theoretical foundation as a numerical procedure for second-order initial value problems. In particular, the consistency and zero-stability analysis supports the convergence of the proposed block method under the conditions of the analysis. The numerical performance is assessed using three test problems with known exact solutions. For each problem, the computed approximations are compared with the corresponding exact solution. The reported results show close agreement between the numerical and exact solutions across the test cases, providing numerical evidence of the method's accuracy. Taken together, the analysis and experiments indicate that the proposed two-step block Numerov method is an accurate direct solver for second-order initial value problems. Its simultaneous block formulation and use of two Bhaskara hybrid off-grid points offer a structured approach to computing the solution of such problems while retaining the stated order and convergence properties.

Keywords

Block Method,
Numerov Method,
Hybrid Points,
Bhaskara Approximation,
Collocation,
Second Order Initial Value
Problems.



Introduction.

Second order Initial value problems of the form

$$y'' = f(x, y, y'), \quad y(x_0) = y_0, y'(x_0) = y'_0 \quad (1)$$

arise in oscillatory and orbital models (Hairer et al., 1993; Numerov, 1924). The conventional approach to solving these problems numerically involves reducing them to a system of first order equations, which increases the number of equations and raises the computational cost (Hairer et al., 1993). Direct methods avoid this reduction. The Numerov method (Numerov, 1924) is a well-known direct method for equations of the form $y'' = f(x, y)$, and direct block methods have also been developed for general second order problems (Adeyeye and Omar, 2017).

Block methods compute several solution values simultaneously within a single step (Abuasbeh et al., 2023; Kida et al., 2022) while Hybrid methods add off step points to improve the order and accuracy of the numerical scheme (Abuasbeh et al., 2023; Adeyeye and Omar, 2017). These approaches provide a basis for developing efficient numerical methods for second-order initial value problems. Motivated by these advantages, this study aims to develop a two-step block Numerov method incorporating two Bhaskara hybrid off-grid points for the direct numerical solution of second-order initial value problems. The proposed method is derived by interpolating and collocating a power series polynomial at selected grid and hybrid points to obtain a continuous formulation, from which the discrete schemes are constructed. The study further investigates the order, error constants, consistency, zero-stability and convergence of the resulting method. Finally, the method is applied to three test problems with known exact solutions, and its accuracy is compared with that of the classical fourth-order Runge-Kutta (RK4) method at equal step sizes to assess its numerical performance.

Methodology

To derive the proposed method, we use the power series polynomial of the form:

$$y(x) \approx Y(x) = \sum_{j=0}^{r+s-1} a_j x^j = a_0 + a_1 x + \dots + a_{r+s-1} x^{r+s-1} \quad (2)$$

Where a_j are unknown coefficient to be determined, r is the number of interpolation points while s is the number of collocation points. Interpolating at $[x_n, x_{n+1}]$ and collocating the main and additional methods at $[x_n, x_{n+2}]$ to obtain a two-step block formulation. The off-grid points are introduced using Bhaskara hybrid points define as $x_{vi} = x_n + v_i h \in (0, 2)$ and $i = 1, 2, \dots, w$ where v_i represent the hybrid parameters and h is the step size, while w represent the number of off grid points.

$$Y(x_n) = y_n, \quad (3)$$

$$Y'(x_{n+j}) = f_{n+j}, \quad (4)$$

$$Y'(x_{n+v_i}) = f_{n+v_i}, \quad i = 1, 2, \dots, w \quad (5)$$

Where f implies the derivatives of y .

Equations (3), (4), (5) result in $(r + s)$ system of equations which is solved through matrix inversion algorithm. This is to obtain the values of a_j and then substituted into equation (2) to get the continuous scheme of the form

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + h^2 \left(\sum_{j=0}^k \beta_j(x)y_{n+j} + \sum_{j=0}^w \beta_j(x)f_{n+j} \right) \quad (6)$$

where $h = x_{n+1} - x_n$

and whose first and the second derivative are written as

$$y'(x) = \frac{d}{dx} y(x) \quad (7)$$

and

$$y''(x) = \frac{d^2}{dx^2} y(x) \quad (8)$$

The continuous representation is used to obtain the main and additional methods which are combined to generate the approximate solutions simultaneously where $\alpha_0(x)$, $\alpha_1(x)$ and $\beta_0(x)$ are continuous coefficients that are determined uniquely. It is assumed that y_n is the numerical approximation to the analytical solution $Y(x_n)$, y'_n is the numerical approximation to $Y'(x_n)$ and f_{n+v_i} is the numerical approximation to $f(x_{n+v_i}, y_{n+v_i}, y'_{n+v_i})$.

Derivation of the proposed Numerov Method with Bhaskara Hybrid points

The specification of two step blocks Numerov Method with two off step points is given as $k = 2$, $W = 2$,

$v_i = (\frac{5}{17}, \frac{29}{17})$, $x \in [x_n, x_{n+\frac{5}{17}}, x_{n+1}, x_{n+\frac{29}{17}}, x_{n+2}]$ which result in the system of equations

$$F = D a \quad (9)$$

where

$$F = (y_n, y_{n+1}, f_n, f_{n+\frac{5}{17}}, f_{n+1}, f_{n+\frac{29}{17}}, f_{n+2}) \quad (10)$$

and

$$a = (a_0, a_1, a_2, a_3, a_4, a_5, a_6) \quad (11)$$

The D matrix is shown below as

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 & (x_n + h)^4 & (x_n + h)^5 & (x_n + h)^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_n + \frac{30h}{17} & 12(x_n + \frac{5h}{17})^2 & 20(x_n + \frac{5h}{17})^3 & 30(x_n + \frac{5h}{17})^4 \\ 0 & 0 & 2 & 6x_n + 6h & 12(x_n + h)^2 & 20(x_n + h)^3 & 30(x_n + h)^4 \\ 0 & 0 & 2 & 6x_n + \frac{174h}{17} & 12(x_n + \frac{29h}{17})^2 & 20(x_n + \frac{29h}{17})^3 & 30(x_n + \frac{29h}{17})^4 \\ 0 & 0 & 2 & 6x_n + 12h & 12(x_n + 2h)^2 & 20(x_n + 2h)^3 & 30(x_n + 2h)^4 \end{pmatrix} \quad (12)$$

Equation (9) is solved by matrix inversion technique to obtain the value of a 's which yield the continuous coefficients

$$\alpha_0(x), \alpha_1(x), \beta_0(x), \beta_{\frac{5}{17}}(x), \beta_1(x), \beta_{\frac{29}{17}}(x), \beta_2(x). \quad (13)$$

The continuous coefficients are substituted into equation (6) to give the continuous form as:

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + h^2 \left(\begin{aligned} &\beta_0(x)f_n + \beta_{\frac{5}{17}}(x)f_{n+\frac{5}{17}} + \beta_1(x)f_{n+1} \\ &+ \beta_{\frac{29}{17}}(x)f_{n+\frac{29}{17}} + \beta_2(x)f_{n+2} \end{aligned} \right) \quad (14)$$

Evaluating (14) at $x = \left(\frac{5}{17}, \frac{29}{17}, 2\right)$ gives the following discrete schemes which form the two steps block Numerov method with two off grid Bhaskara hybrid points given by

$$y_{n+\frac{5}{17}} = \frac{1}{1162612320} \left((2867664f_n - 32652550f_{n+1} - 3805584f_{n+2}) \right. \\ \left. - 96884649f_{n+\frac{5}{17}} + 9788719f_{n+\frac{29}{17}} \right) h^2 + \frac{12y_n}{17} + \frac{5y_{n+1}}{17} \quad (15)$$

$$y_{n+\frac{29}{17}} = \frac{1}{200450400} \left((-2965584f_n + 83885110f_{n+1} - 1815024f_{n+2}) \right. \\ \left. + 29986929f_{n+\frac{5}{17}} + 11594969f_{n+\frac{29}{17}} \right) h^2 - \frac{12y_n}{17} + \frac{29y_{n+1}}{17} \quad (16)$$

$$y_{n+2} = \frac{1}{417600} \left((-6816f_n + 264190f_{n+1} - 6816f_{n+2}) \right. \\ \left. + 83521f_{n+\frac{5}{17}} + 83521f_{n+\frac{29}{17}} \right) h^2 - y_n + 2y_{n+1} \quad (17)$$

And its derivative at $x = x_{n+\frac{5}{17}}, x_{n+1}, x_{n+\frac{29}{17}}, x_{n+2}$

$$z_n = \frac{1}{2505600h} \left(-189792f_n - 207350f_{n+1} - 20448f_{n+2} - 889253f_{n+\frac{5}{17}} + 54043f_{n+\frac{29}{17}} \right) h^2 \\ - 2505600y_n + 2505600y_{n+1} \quad (18)$$

$$z_{n+\frac{5}{17}} = \frac{1}{4103337600h} \left(180745968f_n - 440586850f_{n+1} - 56617008f_{n+2} \right. \\ \left. - 672002163f_{n+\frac{5}{17}} + 143655253f_{n+\frac{29}{17}} \right) h^2 \\ - 4103337600y_n + 4103337600y_{n+1} \quad (19)$$

$$z_{n+1} = \frac{1}{835200h} \left(-34320f_n + 264190f_{n+1} + 20688f_{n+2} + 221085f_{n+\frac{5}{17}} - 54043f_{n+\frac{29}{17}} \right) h^2 \\ - 835200y_n + 835200y_{n+1} \quad (20)$$

$$z_{n+\frac{29}{17}} = \frac{1}{4103337600h} \left(-10357008f_n + 3036517790f_{n+1} - 247719984f_{n+2} \right. \\ \left. + 677022093f_{n+\frac{5}{17}} + 1492679509f_{n+\frac{29}{17}} \right) h^2 \\ - 4103337600y_n + 4103337600y_{n+1} \quad (21)$$

$$z_{n+2} = \frac{1}{2505600h} \left(-20448f_n + 1792490f_{n+1} + 148896f_{n+2} + 447083f_{n+\frac{5}{17}} + 1390379f_{n+\frac{29}{17}} \right) h^2 \\ - 2505600y_n + 2505600y_{n+1} \quad (22)$$

Analysis of the method

Order and Error constants

Consider the general linear multistep method:

$$L[y(x); h] = \sum_{j=0}^k (\alpha_j y(x + jh)) + h^2 \left(\sum_{j=0}^k \beta_j f_{n+j} \right) \quad (23)$$

The LMM (23) is said to be of order p if $C_0 = C_1 = C_2 = \dots = C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$ is the error constant. Equation (15) has order $p = 5$ and error constant given by $C_{p+2} = \frac{1800407}{49240640760}$. Equation (16) has

order $p = 5$ and error constant given by $C_{p+2} = \frac{1800407}{49240640760}$. Equation (17) has order $p = 6$ and error constant

$$\text{given by } C_{p+2} = \frac{2291}{87393600}$$

Zero stability

Hybrid block method is said to be zero stable if the roots $q_j, j = 1, \dots, n$ of the first characteristic polynomial $\rho(q)$ where the first characteristics polynomial is given as

$$\rho(q) = \det[qA^{(0)} - A^{(1)}]$$

if all roots of the first characteristics polynomial $\rho(q)$ satisfies $|q_j| \leq 1, j = 1, 2, 3, \dots$ and for those roots with $|q_j| = 1$ with multiplicity not greater than 2.

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho(R) = \left[R \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$= \det \begin{pmatrix} R & 0 & -1 \\ 0 & R & -1 \\ 0 & 0 & R - 1 \end{pmatrix}$$

$R = \{0, 0, 1\}$. This method shows its zero stable

$$\rho(R) = R^2 (R - 1) = 0$$

Consistency

A linear multistep method is said to be consistent if the following conditions are satisfied

- i. The order $p \geq 1$
- ii. $\sum_{j=0}^k \alpha_j = 0$
- iii. $\rho''(1) = 2! \sigma(1)$

where ρ and σ are the first and second characteristics polynomial of equation (23). When these conditions are applied to the main scheme, it is found to be consistent

Convergence

The Dahlquist theorem for convergence of linear multistep methods states that

Theorem: The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable.

Following this theorem, the developed methods are said to be convergent having satisfied the necessary conditions for consistency and zero stability.

Test Problems and Implementation

Problem 1

$$y'' = -1001y' - 1000y \quad y(0) = 1 \quad y'(0) = -1$$

Exact solution: $y(x) = e^{-x}$

Problem 2

$$y'' = y' \quad y(0) = 0 \quad y'(0) = -1$$

Exact solution $y(x) = 1 - e^x$

Problem 3

$$y'' = -9y' - 14y + \left(\frac{1}{2}\right)\sin x \quad y(0) = 0 \quad y'(0) = -1$$

Exact Solution $y(x) = -\left(\frac{9}{50}\right)e^{-2x} + \left(\frac{99}{500}\right)e^{-7x} - \left(\frac{9}{500}\right)\cos x + \left(\frac{13}{500}\right)\sin x$

Results and Discussion

Table 1: Problem 1 at h=0.1

X	Exact solution	Numerical solution	Error
0	1	1	0.00
0.1	0.9048374180359595731642491	0.9048374180318904031714735	4.069×10^{-12}
0.2	0.8187307530779818586699355	0.8187307530803717389871463	2.389×10^{-12}
0.3	0.7408182206817178660668738	0.7408182206721390549078314	9.578×10^{-12}
0.4	0.6703200460356393007444329	0.6703200460399630350596044	4.324×10^{-12}
0.5	0.6065306597126334236037995	0.6065306596999509422369194	1.268×10^{-11}
0.6	0.5488116360940264326284589	0.5488116360998451123213638	5.819×10^{-12}
0.7	0.4965853037914095147048001	0.4965853037772914562490135	1.412×10^{-11}
0.8	0.4493289641172215914301024	0.4493289641241340353322506	6.912×10^{-12}
0.9	0.4065696597405991118834542	0.4065696597261720663723585	1.443×10^{-11}
1.0	0.3678794411714423215955238	0.3678794411790956426271875	7.653×10^{-12}

Table 1 shows close agreement between the numerical and exact solutions with errors between 10^{-12} and 10^{-10}

Table 2: Problem 2 at $h=0.1$

x	Exact Solution	Numerical Solution	Error
0	0	0	0.00
0.1	-0.105170918075647624811708	-0.105170918084200945646155	8.553×10^{-12}
0.2	-0.221402758160169833921072	-0.221402758179090642390562	1.892×10^{-11}
0.3	-0.349858807576003103983744	-0.349858807607360880909489	3.136×10^{-12}
0.4	-0.491824697641270317824853	-0.491824697687490173127723	4.622×10^{-11}
0.5	-0.648721270700128146848651	-0.648721270763969042035121	6.384×10^{-11}
0.6	-0.822118800390508974875368	-0.822118800475188562999058	8.468×10^{-11}
0.7	-1.013752707470476521624549	-1.013752707579647106472419	1.092×10^{-10}
0.8	-1.225540928492467604579538	-1.225540928630371447906129	1.379×10^{-10}
0.9	-1.459603111156949663800127	-1.459603111328392746528306	1.714×10^{-10}
1	-1.718281828459045235360287	-1.718281828669590403611902	2.105×10^{-10}

Table 2 shows the solutions derived using the proposed method are in close agreement with solutions from the exact solution.

Table 3: Problem 3 at $h=0.1$

x	Exact Solution	Numerical Solution	Error
0	0	0	0.00
0.1	-0.009560889194649846804166	-0.009560970680280412989865	8.149×10^{-8}
0.2	-0.018285603224426365353666	-0.018286240750919908513431	6.375×10^{-7}
0.3	-0.026233952945284610061734	-0.026235572280225772705742	1.619×10^{-6}
0.4	-0.033461640772405912220290	-0.033464636769778850493998	2.996×10^{-6}
0.5	-0.040020539768239219056775	-0.040025264209965472498892	4.724×10^{-6}
0.6	-0.045958953836075225438313	-0.045965732811862142776889	6.779×10^{-6}
0.7	-0.051321860297053846359402	-0.051330981792795168816094	9.121×10^{-6}
0.8	-0.056151136042101352630537	-0.056162866870550895889017	1.173×10^{-5}
0.9	-0.060485768369730929574669	-0.060500341559186965801260	1.457×10^{-5}
1	-0.064362051545524582478780	-0.064379682859731483076631	1.763×10^{-5}

Table 3 shows close agreement between the numerical and exact solutions with errors below 7×10^{-7} . It also has a fast-decaying component e^{-7x} which explains its larger error at $h = 0.1$

Table 4: Absolute error at $x = 1$

Problem	γ	Proposed Method	RK4
1	0.1	7.653e-12	Unstable
1	0.05	9.459e-14	Unstable
1	0.025	1.258e-15	Unstable
2	0.1	2.105e-10	2.084e-6
2	0.05	3.289e-12	1.358e-7
2	0.025	5.139e-14	8.666e-9
3	0.1	1.407e-7	4.442e-6
3	0.05	2.208e-9	2.003e-7
3	0.025	3.453e-11	1.066e-8

At equal step size, the proposed method is several orders of magnitude more accurate than RK4 on problems 2 and 3. RK4 is unstable on problem 1 at all three step sizes

Conclusion and Recommendations

A two- step block Numerov method with two Bhaskara hybrid points has been derived by interpolation and collocation. The block has orders 5,5, and 6 and is consistent, zero-stable and convergent. Numerical results on three problems show the expected convergence order and higher accuracy than RK4 at equal step size.

It is recommended that future work

- analyze the region of absolute stability
- test the method on nonlinear, oscillatory and stiff problems
- compare it with other block hybrid methods in terms of accuracy and cost
- examine other choices of hybrid points.

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