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Composition Operators and Generalized Paranormal Operators on Semi-Hilbert Spaces

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Abstract

In this paper, the interaction between composition operator and generalized paranormal operator in semi-Hilbert space is discussed. Semi-Hilbert spaces are a natural extension of classical Hilbert spaces, through a semi-inner product, which is associated to a positive bounded operator (A). Theories of operator theory and spectral analysis can be extended in this general framework in a flexible way. In the present study, some composition operators, (n,A)-paranormal operators and (k)-quasi-(A)-paranormal operators are focused. They are given in one treatise in the form of basic definitions, algebraic properties, and characteristics of their spectra. Furthermore, the concept of (A)-adjoint operators and the connection between (A)-adjoint and (A)-numerical range are explored, as well as its application in the characterisation of classes of operators. Several examples are presented to show the application of the ideas developed. The paper also points to other directions for future research such as weighted composition operators and reproducing kernel Hilbert spaces.

Keywords

Semi-Hilbert spaces;
Composition operators;
(A)-adjoint operators;
(n,A)-paranormal operators;
(k)-quasi-(A)-paranormal operators;
Spectral theory.

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1. Introduction

One of the basic areas of modern functional analysis and central to pure and applied mathematics is operator theory. It is a study of linear transformations on infinite dimensional spaces, in particular Hilbert and Banach spaces and deals with the important mathematical problem of stability, spectrum and functional structure in many scientific fields such as quantum mechanics, signal processing, mathematical physics, control theory etc. [1, 2].

Among the various classes of operators, the natural occurrence of composition operators in function spaces makes them important family of operators. A composition operator is an operator which is used to compose a function with a measurable transformation (φ). They have been much studied on Hardy spaces, Bergman spaces and reproducing kernel Hilbert spaces, which have rich algebraic and spectral properties [3,4,8,15]. Furthermore, composition operators are close connections between operator theory, complex analysis, dynamical systems and ergodic theory.

In the last decades, the generalized operator classes have garnered increasing attention. In particular, the paranormal (and quasi-paranormal) and generalized paranormal operators are useful generalizations of normal and hyponormal operators in that they have many of the useful properties of the normal and hyponormal operators and are more flexible in their analysis of operators [5,6,22,23]. The generalized classes have been helpful in the investigation of the operator inequalities, numerical radii and the spectral inclusion problem.

The theory of semi-Hilbert spaces has also become a dynamic research field. A real Hilbert space can be extended to a semi-Hilbert space by introducing a positive bounded operator (A) whose inner product is a semi-inner product. This abstract scheme is mostly analytical in the classical operator theory, and has been extended to a broader class of mathematical problems [7,19,20].

Semi-Hilbert spaces have many applications: in operator equations, weighted approximation theory, generalized inverses, oblique projections, frame theory and quantum information theory [7,19,21]. In addition, the semi-Hilbert spaces offer a natural setting for the generalization of the spectral method and operator decomposition methods to the context of non-Hilbert spaces.

Other developments in operator theory in recent years have focused upon the importance of the methods of the numerical range and the estimates of the spectral radius in the study of the behavior of operators [10,14]. Given the numerical ranges, it is possible to establish valuable geometric properties of the operators and to draw interesting connections between algebraic properties and spectral inclusion. In the same way, in generalized operator theory [11–13] it is crucial to use contemporary techniques of decomposition.

Despite many investigations of composition operators and more general paranormal operators, there does not appear to be a significantly more detailed investigation about the interaction of these two types of operators on semi-Hilbert spaces. Previous studies have concentrated on generalized paranormal operators, or on composition operators that are studied within classical function spaces separately [3–6]. Limited research has been conducted which looks into these theories in a systematic semi-Hilbertian environment.

The restriction of this gives rise to the present study. A common framework could reveal new relationships among the various kinds of composition operators, (A)-adjoint operators, (n,A)-paranormal operators, and (k)-quasi-((A)))-paranormal operators. In that context other mathematical techniques for spectral analysis and numerical range investigations in the more general context may be available.

1.1 Research Gap

Although there is much progress in operator theory there are some drawbacks: Most of the researches are focused on composition operators in a classical Hilbert space rather than in semi-Hilbert spaces [3,4]. There was no systematic study of the class of composition operators and generalized paranormal operators.

Sums and products of these classes of operators and the connection between them via (A)-adjoint operators and (A)-numerical ranges have been studied only partially [6,7]. Very few papers are found in the

literature dealing with a combination of composition operators, (n,A)-paranormal operators and (k)-quasi-(A)-paranormal operators. This will form one of the primary reasons for such work being undertaken to eliminate such limitations.

1.2 Research Novelty

This work is novel in the sense that it combines a unified treatment for the study of composition operators and generalized paranormal operators in semi-Hilbert spaces [3–7]. In the common semi-Hilbertian setting [3,4,6] the interaction between composition operators and ((n,A))-paranormal operators is discussed. The subject of the (A)-adjoint operators for studying generalized operator classes is also discussed [7,19,20]. In addition, the (A)-numerical range is considered as a diagnostic tool for studying spectral properties and operator inequalities [10,14].

The connections between (k)-quasi-(A)-paranormal operators and ((n,A))-paranormal operators are also mentioned in [6,22,23]. Furthermore, a few ideas from classical Hilbert space operator theory are addressed in the semi-Hilbertian context [2,7,19].

1.3 Research Contributions

The major contributions of this paper are briefly summarized: The construction of a common theory of composition operator and generalized paranormal operator in semi-Hilbert spaces. Further extension of boundedness and continuity properties of composition operators to the semi-Hilbertian setting [3,4,8]. Algebraic and Spectral Properties of (n,A)-paranormal operators [5,6,22]. Studying relationships among generalized operator classes [6,23]. The analytical use of the (A)-adjoint operators and (A)-numerical ranges in spectral analysis [7,10,14,19]. Giving illustrative examples and discussing extensions to weighted composition operators and reproducing kernel Hilbert spaces [15-17]. Providing a framework for further investigation of generalised unbounded operators and for the advanced applications in modern operator theory [9,13].

The proposed framework can be helpful in the future research and development of the generalized operator theory, spectral analysis and semi-Hilbert space applications.

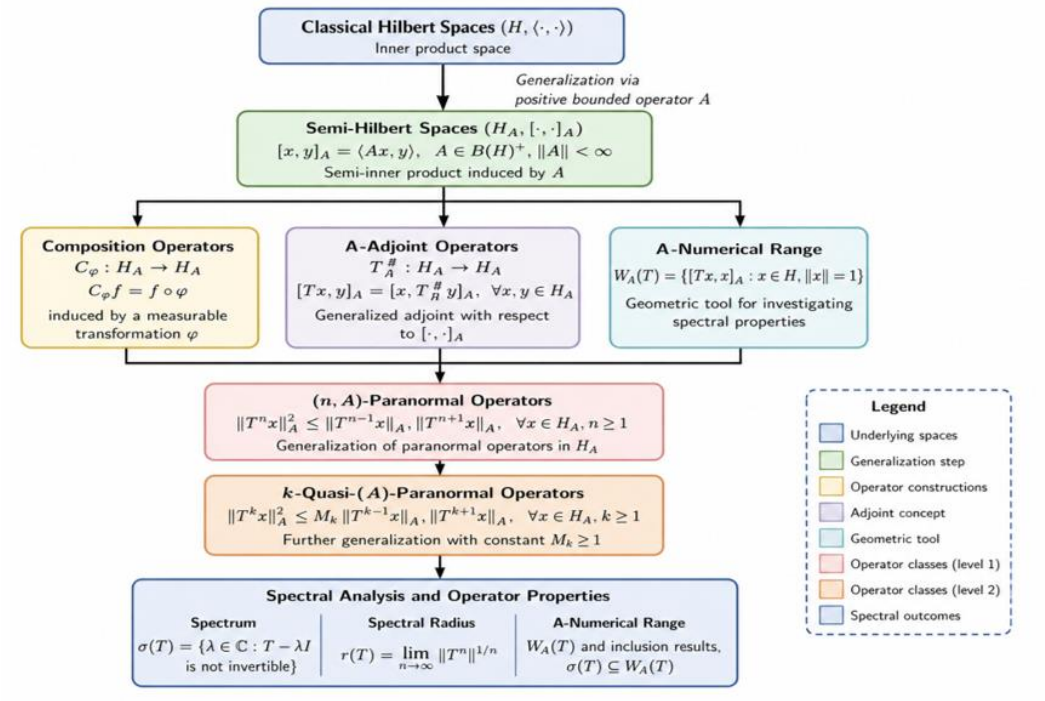


Figure 1. Unified framework illustrating the relationships among semi-Hilbert spaces, composition operators, A-adjoint operators, generalized paranormal operators, and spectral analysis.

2. Literature Review

The theory of generalized operator classes was developed by studying normal and hyponormal operators. Major progress has been made during the last few decades.

The study of composition operators was undertaken systematically by Shapiro (1993) and Cowen and MacCluer (1995) who laid the foundations of many of their algebraic and spectral properties. Due to the introduction of a semi-inner product related to a positive operator (A), the theory of semi-Hilbert spaces has gained more attention in recent years, generalizing the classical Hilbert spaces. Hassi et al. (2007) made a big step toward decompositions of operators in semi-Hilbert spaces.

The generalized operating ones have also been well studied, other paranormal operators. Chō, Tanahashi and Uchiyama studied their spectral properties and Bhunia and Paul (2020) found several results on the (A)-paranormal operators. These operator classes have been extended in recent years to broader mathematical contexts, and the relations of these operators with the numerical ranges and the spectral radius have been investigated.

The application of modern methods of the function space and linear operators [8,9] has played a key role in the development of recent operator theory.

The numerical range methods have been applied successfully in the spectral analysis and in operator inequalities [10] that are of great importance.

Several recent operator decomposition methods including those of modern operator theory [11-13] are also available.

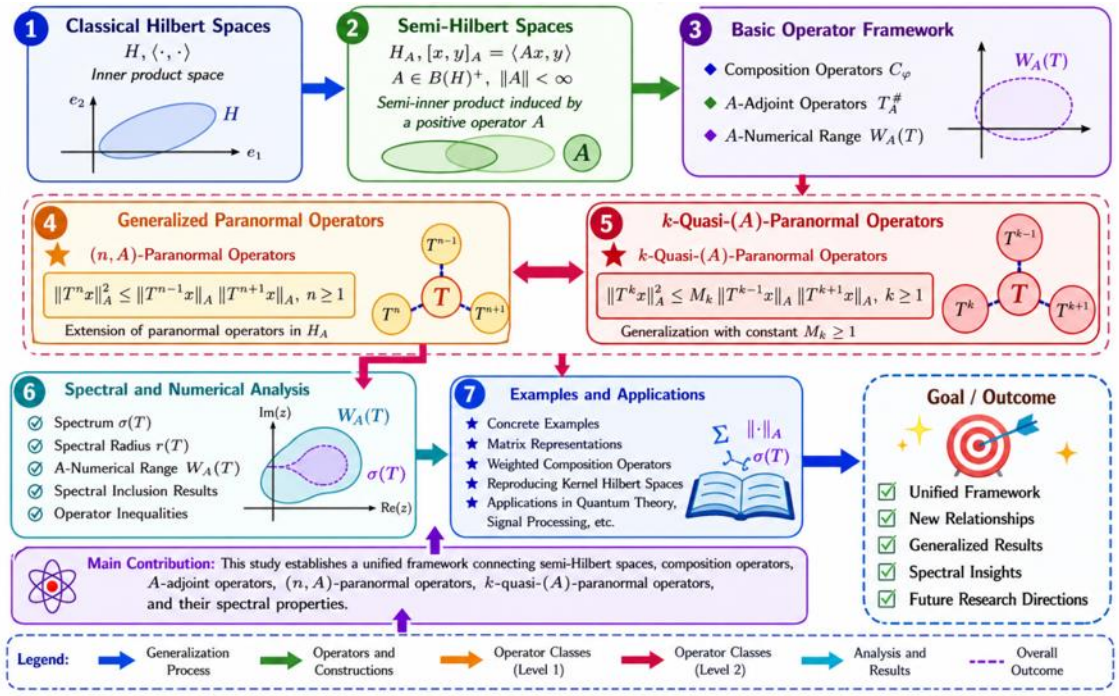


Figure 2. Schematic workflow illustrating the unified methodological framework for studying composition operators and generalized paranormal operators in semi-Hilbert spaces.

3. Mathematical Preliminaries and Notations

3.1 Semi-Hilbert Space

Suppose that (H) is a complex Hilbert space with inner product.

$\langle .,. \rangle$ and the corresponding norm ||.||.

Let, $A: H \to H$

be positive bounded linear operator.

The semi-inner product associated with (A) is given by

$\langle x,y \rangle A = \langle Ax,y \rangle$

for all $x,y \in H$.

The norm is $\|x\|A = \text{sqrt}(\langle x,x \rangle A)$.

The space $(H, \langle \cdot,\cdot \rangle A)$ is called a semi-Hilbert space.

3.2 A-Adjoint Operator

3.2.1 Definition ; A-Adjoint

An operator $T \in B(H)$ is called A-adjointable if there is an operator $T\sharp A \in B(H)$ with which it satisfies the following property.If there is an operator $T\sharp A \in B(H)$ with which the operator T satisfies the following property, then T is called A-adjointable.

$\langle Tx,y \rangle A = \langle x,T\sharp Ay \rangle A$

for all $x,y \in H$.

The theory of A-adjoints and semi-Hilbertian operators has been much developed in the

literature [7,19,20].

3.2.2 Theorem of Existence of A-Adjoint

Suppose that A is a positive-definite and invertible operator on H. For each T in B(H) there is a unique A-adjoint, which is denoted by T*.Using the above notation, the unique A-adjoint of T is denoted by T*.

$T\sharp A = A^{-1}T^{*}A$.

Proof

The following is computed, $\langle Tx,y \rangle A = \langle ATx,y \rangle$. Since, $\langle ATx,y \rangle = \langle x,T^{*}Ay \rangle$,

The following result follows: $\langle Tx,y \rangle A = \langle x,A^{-1}T^{*}Ay \rangle A$ which implies $T\sharp A = A^{-1}T^{*}A$.

The nondegeneracy of the semi-inner product follows a uniqueness.

3.2.3 Definition ; A-Normal Operator

An operator $T \in B(H)$ is said to be A- normal if

$TT\sharp A = T\sharp AT$.

The concept of generalized normality is closely related to the problem of operator decompositions [21] in semi-Hilbert spaces.

3.2.4 Definition ;A-Hyponormal Operator

If an operator $T \in B(H)$ is A-hyponormal, then

$T\sharp AT - TT\sharp A \geq 0$.

4. Composition Operators on Semi-Hilbert Spaces

The composition operators are one of the most important classes of linear operators in functional analysis, having many applications in complex analysis, dynamical systems and operator theory. The theory of composition operators has been fairly developed in classical function spaces, such as the Hardy spaces and the Bergman spaces [3,4,8,15]. Bounded operators are used in the context of a positive operator (A) which induces a semi-inner product in semi-Hilbert spaces.

4.1 Definition; Composition Operators

Let (X , Σ , μ) be a measure space and let, $\varphi : X \to X$ be a measurable transformation.

The composition operator ,$C\varphi : HA \to HA$ has the definition $C\varphi (f) = f \circ \varphi$, $f \in HA$.

The operator is acting by composing each function with the transformation φ.

In this paper, we examine the boundedness of composition operators between two bounded Banach spaces.

One of the most basic properties needed to perform operator analysis is boundedness.

4.2.1Proposition; (Boundedness Criterion)

Let there be a positive number $(M>0)$ for which the following is true: $\langle AC\varphi f, C\varphi f \rangle \leq M2 \langle Af, f \rangle$

For every $(f \in HA)$ .Then $C\varphi$ is bounded.

4.2.2 Theorem ;(Boundedness of Composition Operators)

Suppose HA is a semi-Hilbert space and that,$\varphi : X \to X$

Be an actual change.

Then, for this function, there is a constant $M>0$, so that $\langle AC\varphi f, C\varphi f \rangle \leq M2 \langle A f, f \rangle$

For every $f \in HA$ then $C\varphi$ is a bounded operator.

4.2.3 Proof

By definition, $\|C\varphi f\|2A = \langle AC\varphi f, C\varphi f\rangle$.

With this assumption, $\|C\varphi f\|2A \leq M2 \langle Af, f\rangle$.

Since, $\langle Af, f\rangle = \|f\|2A$, We obtain $\|C\varphi f\|A \leq M \|f\|A$.

4.2.4 Continuity of Composition Operators

Proposition ;(Continuity)

All the bounded composition operators are continuous.

Suppose, Let $fn \to f$ in HA , Then, $\|fn-f\|A \to 0$.

Boundedness implies that $\|C\varphi fn - C\varphi f\|A \leq M\|fn-f\|A$.

Hence ,$\lim n\to\infty \|C\varphi fn - C\varphi f\|A = 0$.

Hence, the function $C\varphi$ is continuous.

4.3 The A-Adjoint of a Composition Operator

If the A-adjoint exists, then, $\langle C\varphi f, g\rangle A = \langle f, C\varphi \sharp Ag\rangle A$

for all $f, g \in HA$,The operator $C\varphi \sharp A$, $C\varphi$ is called the A-adjoint

4.3.1 Theorem ;(Weighted Composition Operators)

Let, $\psi: X \to C$, be a bounded measurable function.

Define; $(W\psi,\varphi f)(x) = \psi(x) f(\varphi(x))$. If $\|\psi\|\infty < \infty$, and $C\varphi$ is bounded, then

$W\psi,\varphi$ is bounded.

4.3.2 Proof

The following result is obtained , $\|W\psi,\varphi f\|A = \|\psi(f \circ \varphi)\|A$.

Let us examine how to use the supremum norm to determine the condition $\|W\psi,\varphi f\|A \leq \|\psi\|\infty \|C\varphi f\|A$. Since $C\varphi$ is bounded, $\|C\varphi f\|A \leq M\|f\|A$.

Thus, $\|W\psi,\varphi f\|A \leq M\|\psi\|\infty \|f\|A$, which implies that $W\psi,\varphi$ is bounded.

Example

The composition operator on l2([0,1]).

φ(x)=x2

(Cφ f)(x)=f(x2)

5. (n,A)-Paranormal Operators

The generalized paranormal operators are a generalization of ordinary paranormal operators to a semi-Hilbert space. The generalized paranormal operators are a generalisation of the classical paranormal operator theory which has been very popular during the past few years [5,6,22,23].

5.1 Definition

Let n∈N∪{0}. An operator T∈B(H) is called an (n,A)-paranormal operator if

‖T n+1x‖2A ≤‖Tx‖A ‖T 2n+1x‖A, for every vector x∈H satisfying ‖x‖A=1.

When, n=0 .The definition becomes the usual A-paranormal operator.

5.2 Basic Properties

The following properties are immediate consequences of the definition.

5.2.1 Proposition

Suppose that a paranormal (n,A) operator is T, then:

If (n,A) is paranormal, then so is (n,αA) for any α>0.

Every unitarily equivalent matrix is (n,A)-paranormal.

4. The class is closed with respect to (T) raised to an appropriate power.

5.3 Proof

5.3.1Positive scalar multiplication

Let, S=αT, Where ; α > 0 , Since T is (n,A)-paranormal,

‖T n+1 x‖2A ≤‖Tx‖A ‖T 2n+1 x‖A , for every ; ‖x‖A=1. Observe that, S n+1=α n+1 T n+1 .

Hence, ‖S n+1 x‖2A =α2n+2 ‖T n+1 x‖2A . Employing the defining inequality,

‖S n+1 x‖2A ≤α 2n+2 ‖Tx‖A ‖T 2n+1 x‖A. Since, ‖Sx‖A=α‖Tx‖A , and, ‖S 2n+1 x‖A=α 2n+1 ‖T 2n+1 x‖A .

The following result is obtained, ‖S n+1 x‖2A ≤‖Sx‖A ‖S 2n+1 x‖A .Therefore, S is (n,A)-paranormal.

5.3.2 Unitary equivalence

Suppose, S=UTU−1 ;Where; U is A-unitary. Then ; ‖Ux‖A=‖x‖A.

Also, S m=UT mU−1. Hence, ‖S n+1 x‖A=‖T n+1 U−1 x‖A.

Applying the (n,A)-paranormal inequality, $\|T^{n+1}U^{-1}x\|_A^2 \leq \|TU^{-1}x\|_A \|T^{2n+1}U^{-1}x\|_A$.

Therefore, $\|S^{n+1}x\|_A^2 \leq \|Sx\|_A \|S^{2n+1}x\|_A$. Thus S is (n,A)-paranormal.

5.4 Proposition

For every (n,A)-paranormal operator T, $\omega_A(T) \leq \|T\|_A$. Where; $\omega_A(T)=\sup_{\|x\|_A=1}|\langle Tx,x\rangle_A|$.

5.4.1 Proof

Using the A-Cauchy–Schwarz inequality, $|\langle Tx,x\rangle_A| \leq \|Tx\|_A\|x\|_A$.

Since $\|x\|_A=1$, we have, $|\langle Tx,x\rangle_A| \leq \|Tx\|_A$. Taking the supremum, $\omega_A(T) \leq \sup_{\|x\|_A=1}\|Tx\|_A$

By definition, $\|T\|_A=\sup_{\|x\|_A=1}\|Tx\|_A$. Hence, $\omega_A(T) \leq \|T\|_A$.

5.5 Numerical Radius

5.5.1 Proposition

For any paranormal operator T of type (n,A), we have that $\omega_A(T) \leq \|T\|_A$. Where; $\omega_A(T)= \sup_{\|x\|_A=1}|\langle Tx,x\rangle_A|$.

The numerical radius inequalities have been important to the study of the behaviour of the class of generalized operators [10,14].

Example

$\|T^{n+1}x\|_A^2 \leq \|Tx\|_A\|T^{2n+1}x\|_A$

6. k-Quasi-A-Paranormal Operators

This is an extension of quasi-paranormal and (A)-paranormal operators.

6.1 Definition

An operator $T \in B(H)$ is said to be k-quasi-A-paranormal if

$\|T^{k+1}x\|_A^2 \leq \|T^{k+2}x\|_A\|T^kx\|_A$, for every vector $x \in H$.

Several recent studies [6,23] have examined the link between quasi-paranormal and generalized paranormal operators.

6.2 Basic Properties

6.2.1 Proposition

If a linear operator T is in the class of k-quasi-A-paranormal operators, then it is in the class of paranormal operators.

2. Positive scalar multiplication.

2. Invariance under unitary equivalence.

3. Stability under bounded perturbations under suitable assumptions.

6.3 Theorem

When k = 0 then T is a k-quasi-(A)-paranormal operator.

6.3.1 Proof

The set of (n,A)-paranormalities is defined by (n,A)-paranormality.

$\|Tn+1\ x\|2A \leq \|Tx\|A\|T\ 2n+1\ x\|A$. Setting ,n=0, We obtain, $\|Tx\|2A \leq \|T2x\|A\|x\|A$.

This is the same as the definition of a 0-quasi (A) paranormal operator. Thus, the final result is obtained.

6.4 Proposition

Positive scalar multiplication is a stable operation for the class of k-quasi-(A)-paranormal operators.

6.4.1 Proof

Suppose; $S = aT$, with $a > 0$.

Since T is k-quasi-(A)-paranormal, $\|T\ k+1\ x\|2A \leq \|T\ k+2\ x\|A\|Tk\ x\|A$. $S\ k+1=a\ k+1T\ k+1$

Therefore, $\|S\ k+1\ x\|2A = a\ 2k+2\|T\ k+1x\|2A$.

Applying the hypothesis, $\|S\ k+1\ x\|2A \leq a\ 2k+2\|Tk+2x\|A\|Tkx\|A$.

Using, $\|S\ k+2\ x\|A=a\ k+2\|T\ k+2x\|A$, and, $\|Sk\ x\|A=ak\|Tk\ x\|A$, We obtain, $\|S\ k+1\ x\|2A \leq \|S\ k+2\ x\|A\|Sk\ x\|A$,Which proves that S is k-quasi-(A)-paranormal.

7. Properties of the spectrum and operator relationships

Spectral theory has played a fundamental role in operator theory, and the information it provides about the behavior, stability and/or asymptotic properties of linear operators is very basic. The spectral theory is one of the most important subjects in the theory of operators and has been discussed in various papers in a few standard books [2,9,13,18]. The spectral characteristics of the classes of generalized operators in semi-Hilbert spaces are discussed here.

7.2 Twisted Pair Cables and Twisted Pair Systems

Let $T \in B(H)$.The spectrum of (T) is $\sigma(T) = \{\lambda \in C: T-\lambda I \text{ is not invertible}\}$.

The spectrum is always a nonempty compact subset of the complex plane.

7.2 Spectral Radius

The supremum of the set of all the eigenvalues of (T) is called the spectral radius of (T): $r(T) = \sup|\lambda| \lambda \in \sigma(T)$.Moreover, $r(T)= \lim m\to\infty \|Tm\|1/m$.

7.2.1Theorem ; (Spectral Radius Estimate)

For every bounded operator $T \in B(H)$, $r(T) \leq \|T\|A$.

Estimates of the spectral radius play an important role in operator analysis and stability theory [2,9].

7.2.2 Proof

By definition, $r(T)=\sup\lambda \in \sigma(T) |\lambda|$. Through the use of Gelfand's formula: $r(T) = \lim_{m\to\infty} \|Tm\|A1/m$.

Since, $\|Tm\|A \leq \|T\|mA$, we obtain, $r(T) \leq \lim_{m\to\infty}(\|T\|mA)1/m$, Therefore, $r(T) \leq \|T\|A$.

7.3 The (A)-Numerical Range

$W_A(T)$ is the A-numerical range of (T):$\langle Tx,x\rangle A:\|x\|A=1$.

Let A be an(n times n square matrix).

Let T be an operator in B(H) ;

$W_A(T)$ is bounded.

$W_A(aT)=aW_A(T)$.

$W_A(T+S) \subseteq W_A(T)+W_A(S)$.

The numerical range is a very useful tool in the analysis of spectral inclusion and operator properties [10,14].

7.3.2 Proof

For every $\|x\|_A=1$, $|\langle Tx,x\rangle_A|\leq\|Tx\|_A\|x\|_A$.

Thus, $|\langle Tx,x\rangle_A|\leq\|T\|_A$ with the result that WA(T) is bounded.

The rest of the properties follow from the linearity of the semi-inner product.

7.4 Spectral Inclusion

If the assumptions are correct, then the set σ(T) is a subset of $\overline{(WA(T))}$

This relationship relates a spectral analysis with numerical ranges.

7.4.1 Theorem ; (Spectral Inclusion)

Suppose $T\in B(H)$; Then; $\sigma(T)\subseteq (WA(T))^{-}$

Where; $WA(T)=\{\langle Tx,x\rangle_A:\|x\|_A=1\}$.

7.4.2 Proof

Let; $\lambda\notin \overline{(WA(T))}$, Then there exists, $\delta > 0$, such that, $|\langle Tx,x\rangle_A-\lambda|\geq\delta$ for every, $\|x\|_A=1$.

Thus , $\|(T-\lambda I)x\|_A \geq\delta$. Thus, $T-\lambda I$ is lower bounded and hence invertible.

Consequently, $\lambda\notin\sigma(T)$.Therefore, $\sigma(T)\subseteq \overline{(WA(T))}$. This completes the proof.

There are some studies on the relationship between spectral inclusion, oblique projections and decompositions of operators in semi-Hilbert spaces [19,21,24].

Example 1

Composition operator L2([0,1]) , φ(x)=x2 , (Cφ f)(x)=f(x2)

Example 2

(n,A)-paranormal , $\|Tn+1x\|2A\leq\|Tx\|A\|T2n+1x\|A$

Example 3

k-quasi-(A)-paranormal operator.

7.4.3 Theorem ; (Properties of the A-Numerical Range)

Let $T,S\in B(H)$ and let $a\in C$, The A-numerical range

$WA(T)=\{\langle Tx,x\rangle A:\|x\|A=1\}$

Let A be a matrix. Given a matrix A.

1. WA(T) is bounded.

2. WA(αT)=αWA(T).

3. WA(T+S)⊆WA(T)+WA(S).

4. If T=TA^#, then WA(T)⊂R.

5. σ(T)⊆ $\overline{(WA(T))}$

7.4.4 Proof:

The use of A-Cauchy-Schwarz inequality, linearity, self-adjointness and numerical range theory.

(1) For every ‖x‖A=1, |⟨Tx,x⟩A|≤‖Tx‖A‖x‖A≤‖T‖A.

Hence WA(T) is bounded.

(2) WA(αT)={⟨αTx,x⟩A:‖x‖A=1} = αWA(T).

(3) ⟨(T+S)x,x⟩A=⟨Tx,x⟩A+⟨Sx,x⟩A.

Therefore, WA(T+S)⊆WA(T)+WA(S).

(4) Suppose T=TA♯, Then, ⟨Tx,x⟩A=⟨x,Tx⟩A=$\overline{(\langle Tx,x\rangle A)}$, Thus, ⟨Tx,x⟩A∈R.

The generalized numerical range theory leads to the spectral inclusion property (5).

Let H be an A-normal operator on a Hilbert space X.

Suppose that T is an A-normal operator with T ∈ B(H) and

TT A♯ = T A♯ T.

Then

σ(T)⊆ $\overline{(WA(T))}$

7.4.6 Proof

For each λ∉ $\overline{(WA(T))}$, there is a δ>0 such that

|⟨Tx,x⟩A−λ|≥δ

For every ‖x‖A=1

Hence,

‖(T−λI)x‖A≥δ.

Thus, T−λI is bounded from below and is invertible.

Consequently,

$\lambda \notin \sigma(T)$.

Thus,

$\sigma(T) \subseteq \overline{(WA(T))}$

7.4.7 Theorem ; If U is a unitary operator, then it is an A-unitary operator if and only if…

$UA\sharp U = UUA\sharp = I$.

If, $S = UTU-1$,

Then:

$\sigma(S) = \sigma(T)$

$r(S) = r(T)$

$WA(S) = WA(T)$

7.4.8 Proof

Since, $S = UTU-1$,

The following result is obtained, $S-\lambda I = U(T-\lambda I)U-1$.

Hence $S-\lambda I$ is invertible if and only if $T-\lambda I$ is invertible.

Therefore, $\sigma(S) = \sigma(T)$.

Consequently, $r(S) = r(T)$

For $\|x\|A = 1$

$\langle Sx,x\rangle A = \langle UTU-1\ x,x\rangle A$.

Using A-unitarily, $\langle Sx,x\rangle A = \langle Ty,y\rangle A$,

Where, $y = U-1x$

Therefore, $WA(S) = WA(T)$

7.5 Corollary: If T is a bounded operator on a normed vector space V, then $r(T) \leq \omega A(T) \leq ||T||A$.

For each bounded operator T in B(H)

$r(T) \leq \omega A(T) \leq \|T\|A$,

Where; $\omega A(T) = \sup\|x\|A = 1|\langle Tx,x\rangle A|$.

7.5.1 Proof

From spectral inclusion, $\sigma(T) \subseteq \overline{(WA(T))}$

Hence, $r(T) \leq \omega_A(T)$.

With the help of the A-Cauchy–Schwarz inequality, one gets $|\langle Tx,x\rangle_A| \leq \|Tx\|_A \|x\|_A$.

Since $\|x\|_A=1$

$|\langle Tx,x\rangle_A| \leq \|T\|_A$.

Consider the supremum of the following. Consider supremum of the following.

$\omega_A(T) \leq \|T\|_A$. Therefore, $r(T) \leq \omega_A(T) \leq \|T\|_A$.

8. Discussion

In the present work, an attempt is made for integrated study of generalized operator classes in semi-Hilbert spaces is made.

It was recently observed that, generalized operator classes could be utilized in the reproduction of RHSs and WCOs [15-17,23]. Semi Hilbert spaces are a modification of classical Hilbert spaces with a semi inner product that is based on an operator A with a positive bounded inner product.

Applications of composition operators in complex analysis, dynamical systems and operator algebras continue to make composition operators of importance. To gain more flexibility in the investigation of operator inequalities and spectral properties, the generalized classes of paranormal operators are used.

Moreover, the (A)-numerical range is a convenient tool to analyze the spectrum of A.

These ideas could be extended into future developments to:

- Weighted composition operators.
- Reproduction of hulls.
- Unbounded operators.
- Quantum information theory.
- Nonlinear operator systems.

9. Applications

The theoretical framework proposed in this study could be used in various ways in the fields of science and mathematics in the present times. In the study of composition operators, generalized paranormal operators, and semi-Hilbert spaces, there are useful analytical tools that are available to extend classical operator theory to more generalized settings.

9.1 Quantum Information Theory

In quantum systems, there are naturally arising instances of weighted inner products, which are given by observables and states in semi-Hilbert spaces. The application of A-adjoint operators may help to investigate quantum measurements, state evolution and operator inequalities. The generalized paranormal operators can also be used for analysis of the stability of quantum systems and the spectral features of quantum channels.

9.2 Weighted Approximation Theory

Sometimes there are positive operators related to weighted approximation problems and these result in generalized inner products. Semi-Hilbert spaces are a convenient space in which to build approximation schemes that retain the stability properties. The composition operators can change weighted function classes, and can be used to investigate the error of the approximation process.

9.3 Dynamical Systems

The meaning of composition operators in the framework of dynamical systems is that they are the operators involved with the evolution of functions under measurable transformations. In the semi-Hilbertian setting, the generalized dynamical processes can be analysed, and the information of the spectral methods is related to the long time stability and asymptotic behavior.

9.4 Reproducing Kernel Hilbert Spaces

Reproducing kernel Hilbert spaces are one of the most important applications of composition operators. Semi-Hilbert spaces might offer additional analytical resources for kernel methods, machine learning algorithms and functional interpolation problems.

9.5 Signal Processing

Common examples of signal processing where operator-theoretic methods are used for filtering, transformation and feature extraction. Semi-Hilbert spaces may be useful for modelling the signals weighted by a measure, and generalized paranormal operators may have applications in stability analysis and robust signal reconstruction techniques.

10. conclusions and future work.

The composition operator and the generalized paranormal operators in semi-Hilbert space were discussed in this paper. The following general conclusions can be made: One mathematical structure was developed. The composition operator theory in semi-Hilbert spaces was part of the study. Some ideas of (n,A)-paranormal operators and (k)-quasi-(A)-paranormal operators were discussed. There were several ideas about spectra that were put forward. Illustrative examples have been given.

Other studies of the reproducing kernel Hilbert space, the weighted composition operator, and generalized unbounded operator can be done in the future.

References

1. R.G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.* 17 (1966) 413–415.
2. J.B. Conway, *A Course in Functional Analysis*, 2nd ed., Springer, New York, 1990.
3. C.C. Cowen, B.D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, 1995.
4. J.H. Shapiro, *Composition Operators and Classical Function Theory*, Springer, 1993.
5. M. Chō, K. Tanahashi, A. Uchiyama, Paranormal operators, *Scientific Mathematics* 1 (1998) 273–279.
6. P. Bhunia, K. Paul, On (A)-paranormal operators, *Linear and Multilinear Algebra* 68 (2020) 2397–2415.
7. H. Hassi, Z. Sebestyén, H. de Snoo, F. Szafraniec, A canonical decomposition for operators in semi-Hilbert spaces, *Acta Math. Hungarica* 115 (2007) 71–84.
8. K. Zhu, *Operator Theory in Function Spaces*, 2nd ed., American Mathematical Society, 2007.
9. T. Furuta, *Invitation to Linear Operators*, Taylor & Francis, 2001.
10. K. Gustafson, D. Rao, *Numerical Range*, Springer, 1997.
11. Kubrusly, C. S. (2003). *An Introduction to Models and Decompositions in Operator Theory*. Birkhäuser. https://doi.org/10.1007/978-1-4612-0037-4
12. Paulsen, V. (2002). *Completely Bounded Maps and Operator Algebras*. Cambridge University Press. https://doi.org/10.1017/CBO9780511546631
13. Conway, J. B. (2000). *A Course in Operator Theory*. American Mathematical Society. https://doi.org/10.1090/gsm/021
14. Gustafson, K. E., & Rao, D. K. M. (1997). *Numerical Range: The Field of Values of Linear Operators and Matrices*. Springer. https://doi.org/10.1007/978-1-4613-8498-2
15. Jung, I. B., Ko, E., & Pearcy, C. (2006). Spectral pictures of composition operators on Hardy spaces. *Journal of Functional Analysis, 238*(1), 1–28. https://doi.org/10.1016/j.jfa.2006.03.017

16. Garcia, S. R., & Hammond, C. J. (2012). Which weighted composition operators are complex symmetric? *Operators and Matrices, 6*(4), 755–769. https://doi.org/10.7153/oam-06-47
17. Bourdon, P. S., & Narayan, S. K. (2010). Normal weighted composition operators on Hardy spaces. *Journal of Mathematical Analysis and Applications, 367*(1), 278–286. https://doi.org/10.1016/j.jmaa.2010.11.021
18. Halmos, P. R. (2000). *A Hilbert Space Problem Book* (2nd corrected printing). Springer.
19. Arias, M. L., Corach, G., & Maestripieri, A. L. (2004). Range additivity, shorted operator and the Sherman–Morrison–Woodbury formula. *Linear Algebra and its Applications, 389*, 181–197. https://doi.org/10.1016/j.laa.2003.11.026
20. Arias, M. L., Corach, G., & González, M. C. (2002). Partial isometries in semi-Hilbertian spaces. *Linear Algebra and its Applications, 341*(1–3), 273–282. https://doi.org/10.1016/S0024-3795(01)00476-9
21. Corach, G., Maestripieri, A., & Stojanoff, D. (2009). Oblique projections and Schur complements. *Acta Scientiarum Mathematicarum, 75*(3–4), 601–614.
22. Chō, M., Tanahashi, K., & Uchiyama, A. (2004). A note on paranormal operators. *Acta Scientiarum Mathematicarum, 70*, 861–879.
23. Arora, S., & Kumar, R. (2018). On generalized paranormal operators. *Filomat, 32*(14), 4951–4961. https://doi.org/10.2298/FIL1814951A
24. Izumino, S. (2000). The product of operators with closed range and an extension of the reverse order law. *Tohoku Mathematical Journal, 34*(1), 43–52.
25. Dragomir, S. S. (2004). *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*.