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Stability Analysis of Nonlinear Ordinary Differential Equations: Derivations, Comparative Criteria, and Geometric Illustrations; Article Review

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Abstract

The local and global stability problems of the nonlinear ordinary differential equation are discussed in a common mathematical perspective with a critical consideration. These linearization test, direct method of Lyapunov, estimates of exponential decays and the invariance principle of LaSalle are developed and contrasted under the corresponding conditions. Scalar and planar examples are presented that show the difference between stability, attractivity, asymptotic stability and exponential convergence. The theoretical results are complemented with complementary geometric interpretations in the form of analytical phase portraits. Special emphasis is placed on the insufficient stability tests, the interpretation of negative semidefinite Lyapunov derivative, and the extra conditions needed to impose a global stability. The review shows that sometimes a failure of a sufficient condition is not necessarily a sign of instability and convergence cannot be deduced from a non-increasing Lyapunov function. Last, the advantages and disadvantages of the methods investigated are summarized and future research directions involving the use of computational stability certificates, uncertain systems and systematic literature assessment are introduced to further mathematical research and methodological development.

Keywords

Nonlinear ODEs;
Lyapunov Stability;
Linearization;
Lasalle Invariance;
Exponential Stability;
Phase Portraits;
Converse Theorems.

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1. Introduction

Physical, biological, chemical and engineered systems evolve in different ways might be described in a common mathematical language by using nonlinear ordinary differential equations (ODEs). In these models, stability analysis involves examining the question of whether a trajectory starting close to equilibrium will converge to the equilibrium, but also whether it will remain close to the equilibrium, or whether it will converge at a measurable rate. These are all separate questions: Lyapunov stability is about persistence around an equilibrium point, attractivity is about convergence, asymptotic stability is about the persistence and convergence, and the requirement of a uniform decay estimate is added in the case of exponential stability. These classifications are well known and it is found that they depend on the vector field, the equilibrium, and the region where the assumptions are valid [1–4, 21–25].

The linearization method is usually performed early in the diagnostic process due to its ability to linearize a nonlinear system near an equilibrium point using the Jacobian matrix. Under the standard smoothness assumptions, the eigenvalues determine the local behavior when the equilibrium is hyperbolic. The test is not conclusive, however, if eigenvalues are on the imaginary axis, and may not be used to determine global behavior. However, direct Lyapunov methods tackle this limitation by employing a scalar function whose decrease along trajectories proves stability, without needing an explicit solution. In addition, when the derivative of the candidate function is only negative semidefinite, as in the case of LaSalle's invariance principle [1-3, 5-7] this approach becomes stronger.

The second important difference is between the proof of a qualitative property and a certificate of a quantitative property. Asymptotic convergence can be guaranteed if the Lyapunov function decreases strictly, while exponential estimates need comparison bounds involving the Lyapunov function and the derivative of the Lyapunov function. Under appropriate regularity and stability assumptions, converse theorems shed light on the existence of Lyapunov functions but generally do not give a simple constructive formula. These questions foster further research in the development of polynomial certificates, computational Lyapunov functions, and stability under uncertainty/external inputs [3-4, 8-16].

This article is a selective, mathematically oriented review, and not a systematic review of all published work. Its contribution is a unified derivation-led comparison of linearization, the direct Lyapunov method, exponential decay estimates and the LaSalle's invariance principle, with the help of scalar examples and an example of an oscillator system with energy function. The emphasis is on the assumptions that are needed for each of the conclusions, on what kind of tests are sufficient and on the difference between the local and the global claims. Directions for a larger literature survey are included on extensions to switched, delayed, nonsmooth, and input driven systems [2-4, 8-20].

2. Scope and mathematical setting

Suppose that one has the autonomous initial-value problem $\dot{x}=f(x)$, $x(0)=x_0$, where $x\in\mathbb{R}^n$, $f(0)=0$ and f is locally Lipschitz. Local uniqueness is provided by the Lipschitz condition, which requires forward completeness to be proven independently when conclusions are made for all $t\geq 0$. In all that follows, $\|\cdot\|$ will refer to the Euclidean norm. In nonautonomous systems uniformity in initial time isn't automatic but is based upon additional hypotheses.

$$\dot{x}(t) = f(x(t)), \quad f(0) = 0, \quad x(0) = x_0. \tag{1}$$

The origin is stable if given any $\varepsilon>0$, there is a $\delta>0$ so that if $\|x_0\|<\delta$ then $\|x(t)\|<\varepsilon$ for all $t\geq 0$. It is attractive if the initial conditions are sufficiently small to converge to zero and also it should be asymptotically stable.

Exponential stability is defined as: For each $M>1$, there exists $a>0$ such that $\|x(t)\| \leq M \exp(-at)\|x_0\|$ for all $t>0$ in a neighborhood of zero.

3. Linearization: derivation and limits

Suppose that f is continuously differentiable in a neighborhood of 0. Taylor expansion gives $f(x)=Ax+r(x)$, where $A=Df(0)$ and $\|r(x)\|/\|x\| \to 0$ as $x \to 0$. The solution of the linear system $\dot{x}=Ax$ is $x(t)=\exp(At)x_0$. The bounds of the matrix exponential, when A is Hurwitz (all its eigenvalues have strictly negative real parts) suggest $\|\exp(At)\| \leq K \exp(-\beta t)$. [1–4, 21–23]

$$f(x) = Ax + r(x), \quad A = Df(0), \quad \lim[x \to 0] \|r(x)\|/\|x\| = 0. \tag{2}$$

There is a possibility of making the nonlinear implication explicit. If Q is positive definite then $A^TP+PA=-Q$ has a unique positive definite solution P. Take $V=x^TPx$. Then $\dot{V}=x^T(A^TP+PA)x+2x^TPr(x)=-x^TQx+2x^TPr(x)$. This last term is at most $\lambda\min(Q)\|x\|^2 / 2$ for small enough $\|x\|$. Thus, $\dot{V} \leq -\lambda\min(Q)\|x\|^2/2$, which means that it is locally exponentially stable.

$$A^TP + PA = -Q; \quad V=x^TPx; \quad \dot{V} \leq -\tfrac{1}{2} \lambda\min(Q) \|x\|^2. \tag{3}$$

Under normal C^1 conditions, if the real part of the number is positive, then the equilibrium is unstable under the eigenvalue condition. The result of linearization is inconclusive if the eigenvalues are on the imaginary axis and none of them has a positive real part. That is, in the example $\dot{x}=-x^3$ and $\dot{x}=x^3$ have the same Jacobian $A=0$ at $x=0$, but are asymptotically stable and unstable, respectively.

4. Lyapunov direct method: a derivation

Suppose V is continuously differentiable on a neighbourhood D of 0, such that $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$. If a solution is differentiable, then along it, application of the chain rule yields $\dot{V}(x(t)) = (\nabla V(x(t))^T f(x(t))$. Assume that $\dot{V} \leq 0$ in D and choose an arbitrarily small compact sublevel set $\Omega c = \{x:V(x) \leq c\}$ in D. Solutions which begin in Ωc stay in Ωc since V cannot increase. The stability comes from the fact that once we have a δ positive definite and compact, we have an ε-ball that contains the Ωc as well as the initial ball. [1–3, 5–7]

$$dV(x(t))/dt = \nabla V(x(t))^T f(x(t)) \leq 0. \tag{4}$$

Asymptotic stability locally follows if $\dot{V}$ is strictly negative away from the origin.If $\dot{V}$ is strictly negative away from the origin, then asymptotic stability follows locally. If the claim is global, then V and its derivative conditions are required to be global, and V has to be proper (radially unbounded) in which case sublevel sets are compact. If the derivative is negative definite, but the absolute value of it is not large enough, it is not necessarily convergent.

4.1 Exponential estimates from comparison inequalities

Suppose that there exist locally constants $c_1,c_2,c_3>0$ and $p>0$ with the property that $c_1\|x\|^p \leq V(x) \leq c_2\|x\|^p$ and $\dot{V} \leq -c_3\|x\|^p$. Suppose $c_1>0$, $c_2>0$, $c_3>0$ and $p>0$ are locally constant with the property that $c_1\|x\|^p \leq V(x) \leq c_2\|x\|^p$ and $\dot{V} \leq -c_3\|x\|^p$. The multiplication of $\exp((c_3/c_2)t)$ and integration of the above inequality result in $V(t) \leq V(0) \exp(-c_3t/c_2)$, where V(0) is the upper bound. The bounds are combined to get an estimate of the norm that is exponential. [1–4, 8–11]

$$\|x(t)\| \leq (c_2/c_1) \hat{}(1/p) \exp[-c_3t/(p\ c_2)] \|x_0\|. \tag{5}$$

The following argument highlights the difference between the negative derivative and rate-certified derivative: strict negativity does not necessarily mean that the system is asymptotically stable with a uniform exponential rate.

5. Worked scalar counterexamples

5.1 Nonlinear decay without exponential stability

Separation of variables yields, for $\dot{x}=-x^3$, $dx/x^3=-dt$. Integrating from 0 to t yields $-1/(2x(t)^2)+1/(2x_0^2)=-t$, and therefore $x(t)=x_0/\sqrt{(1+2x_0^2 t)}$. So that each solution reduces algebraically to zero. Choosing $V=x^2/2$ gives $\dot{V}=x(-x^3)=-x^4<0$ for $x\neq 0$. However, there is not a single local exponential bound; that is, for any fixed nonzero x_0, the ratio $x(t)/\exp(-at \to \infty$ as $t \to \infty$. [1-3, 21-23]

$$x(t)=x_0 / \sqrt{(1+2x_0^2 t)}, \quad V=x^2/2, \quad \dot{V}=-x^4. \qquad (6)$$

5.2 Instability and multiple equilibria

For $\dot{x}=x-x^3$, equilibria are -1, 0, 1. The derivative $f'(x)=1-3x^2$ has $f'(0)=1>0$, so 0 is unstable; $f'(\pm 1)=-2<0$, locally exponentially stable. The solution to $x = x2$ is $y = 1 - y2$, which is a logistic equation. For $x_0\neq 0$, $x(t)=x_0 e^t/\sqrt{(1+x_0^2(e^{2t}-1))}$. The basin of attraction of the value +1 is $x_0>0$, and of the value -1 is $x_0<0$.

$$x(t)=x_0 e^t / \sqrt{[1+x_0^2(e^{2t}-1)]}. \qquad (7)$$

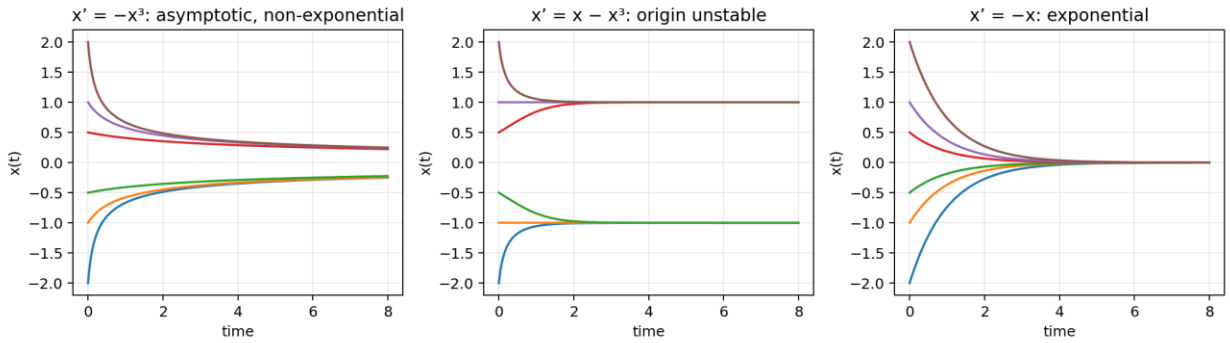


Figure 1. Exact solution trajectories for three scalar equations; all curves are generated from the displayed formulas.

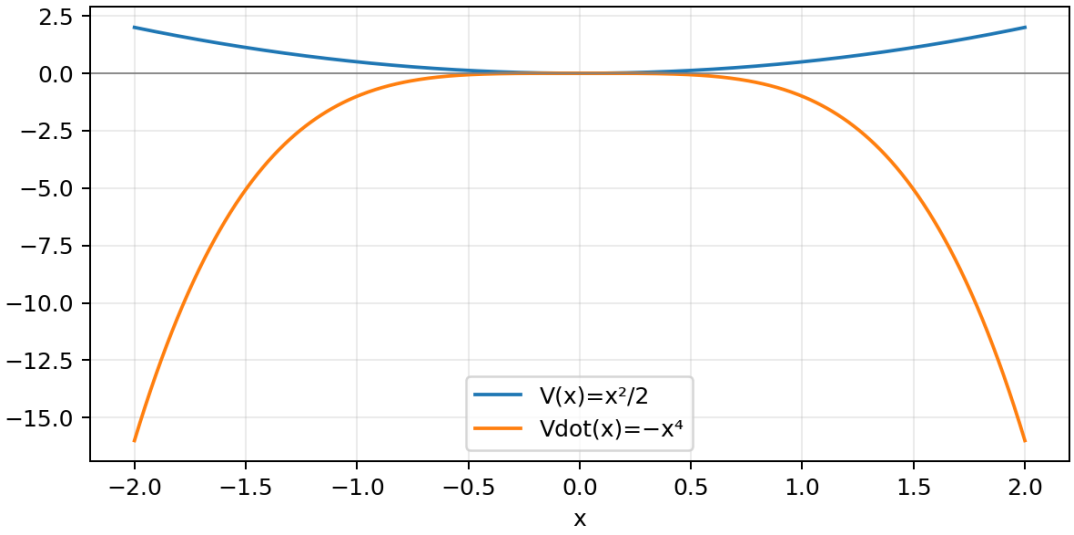


Figure 2. Candidate Lyapunov function and its orbital derivative for $\dot{x}=-x^3$.

6. LaSalle invariance principle and energy methods

Look at the damped nonlinear oscillator $\dot{x}=v$, $\dot{v}=-x-x^3-cv$, where $c>0$. Its natural energy is $V(x,v)=v^2/2+x^2/2+x^4/4$. If you differentiate along the solutions, then $\dot{V}=v(-x-x^3-cv)+(x+x^3)v=-cv^2\leq 0$. The sublevel sets of V are compact and forward invariant due to the radial unboundedness of V. The set where $\dot{V}=0$ is $E=\{(x,v):v=0\}$. If a trajectory remains in E, then it must also have $\dot{v}=-x-x^3=0$, indicating $x=0$. Hence, the origin is the largest invariant subset of E, and the global asymptotic stability follows from LaSalle's principle. [5–7]

$$V=\tfrac{1}{2}v^2+\tfrac{1}{2}x^2+\tfrac{1}{4}x^4, \qquad \dot{V}=-cv^2\leq 0. \tag{8}$$

If c = 0, then there's no loss of energy. Nontrivial trajectories follow level curves and thus are not in the direction of the origin, and so the equilibrium is stable but not asymptotically stable. This comparison makes the point that the additional invariant-set analysis of semidefinite dissipation is precisely the point.

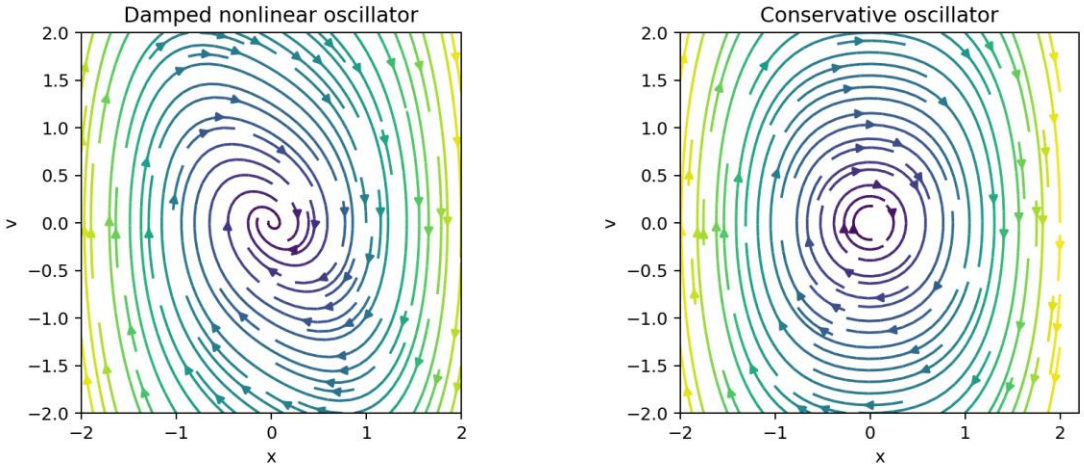


Figure 3. Phase portraits of the damped and conservative nonlinear oscillators; the conservative case has closed energy contours.

7. Comparative mathematical assessment

Table 1. Comparative Summary of Major Stability Analysis Methods for Nonlinear Ordinary Differential Equations

Method	Key assumptions	Typical conclusion	Critical limitation
Linearization	$f\in C^1$; hyperbolic equilibrium	Local exponential stability if A Hurwitz	Inconclusive for zero-real-part eigenvalues
Direct Lyapunov	Positive definite V; derivative sign	Local or global stability / attractivity	Constructing V may be difficult; sufficient not necessary as stated
LaSalle	Compact invariant set; $\dot{V}\leq 0$	Attraction to largest invariant subset	Requires identifying invariant motions in $\dot{V}=0$

Converse Lyapunov	Appropriate stability and regularity hypotheses	Existence of a Lyapunov certificate	Often nonconstructive or hard to compute
Quadratic matrix inequality	A Hurwitz or matrix inequalities	Explicit local exponential bounds	May be conservative for nonlinear/global claims

The methods are not mutually exclusive and they are not presented in a hierarchy. Linearization is affordable and regional; a customized Lyapunov function can achieve global result at the cost of invariant-set analysis; LaSalle lets go of strict decrease for the sake of global result. Converse results provide theoretical existence assertions but don't necessarily provide a computational certificate. [1–4, 8–11].

Table 2. Selected Recent Directions in Nonlinear Stability Analysis and Their Contributions

Research direction	Main contribution	Relevance to this review
Computational Lyapunov functions	Algorithms seek certificates when an analytic Lyapunov function is difficult to construct.	Extends the direct method toward reproducible computational verification [3, 16].
Input-to-state stability (ISS)	Quantifies how bounded disturbances and initial conditions affect state trajectories.	Extends stability analysis beyond autonomous systems [8–12].
Finite-time and fixed-time stability	Studies convergence in finite time or within a uniform settling-time bound.	Adds rate concepts beyond asymptotic and exponential convergence [10].
Control Lyapunov and barrier functions	Connects stability certificates with control design and safety constraints.	Links stability analysis to modern nonlinear-affine control [15].
Switched and interconnected systems	Uses multiple/common Lyapunov functions and small-gain conditions.	Highlights assumptions needed for stability under switching and interconnection [14, 17–18].

8. Extensions and open mathematical questions

Modern stability theory is a development of the classic concepts for switched, delayed, uncertain, hybrid and input-driven systems. Care should be taken of these extensions: A common Lyapunov function for arbitrary switching is stronger than mode wise stability; For delay systems, functionals over histories are used instead of state functions; Input-to-state stability is not a stability property of the state of the system, but a stability property of the input. One of the directions that could be explored via a focused future survey is on constructive certificates, sum-of-squares relaxations of polynomial systems, non-smooth Lyapunov functions, and/or data-driven stability

certificates, all of which could be considered under a similar definition of robustness and computational tractability. [8–20]

Some intriguing research questions that are yet to be answered are: (i) how to quantify conservatism when the dynamics in V are restricted to a polynomial class or a quadratic class; (ii) what additional assumptions allow local certificates to become global; (iii) how to validate computational certificates against numerical rounding and model uncertainty; and (iv) how to modify stability conclusions for nonautonomous dynamics and nonsmoothed dynamics. Whereas these are guidelines for research, not the ones that the literature has left completely unsolved [20-25].

10. Conclusion

The derivations lead to three key lessons. Hurwitz linearization gives an exponential stability locally, but is no solution for nonhyperbolic equilibria. Second, the Lyapunov method transforms the geometric stability in the inequality in the trajectory; quantitative comparison bounds transform that inequality to rates. Third, using the principle of LaSalle, the characterisation of invariant motions in the set of zeros of the derivatives is separated from the set of dissipation. Explicit examples are provided to illustrate the distinction between asymptotic and exponential stability; furthermore, it is shown that negative semidefinite energy dissipation does not guarantee attraction, and does guarantee it [1–3, 5–11].

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