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Numerical Analysis of Finite Element Methods for Nonlinear Problems: Stability and Convergence

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Abstract

We analyze conforming finite element approximations of nonlinear diffusion problems whose radial constitutive flux satisfies uniform monotonicity and Lipschitz bounds. We distinguish general monotone-operator results from their specialization to the constitutive laws studied here. Under the stated assumptions, the discrete solution is quasi-optimal in the energy norm, with a constant depending on the square root of the ratio between the upper Lipschitz and lower monotonicity bounds. A residual-based a posteriori estimate establishes reliability through Galerkin orthogonality and local interpolation. In two spatial dimensions, for polynomial degree \(k \ge 2\), primal and dual regularity assumptions yield an \(L^2\) error estimate of order \(k+1\). For every fixed mesh, an Armijo-damped Newton method converges globally. A separate quadratic Newton error estimate gives a mesh-independent iteration bound when the initial iterate is sufficiently accurate; nested Galerkin solutions provide such starting values under the stated conditions when \(k\) exceeds half the spatial dimension. We compute explicit constitutive bounds for three viscosity laws and identify a case with a contrast ratio of 100. Higher-order numerical results are reported as illustrations of the theoretical estimates, rather than as additional proof. Independent piecewise linear computations reproduce the displayed errors for all three laws. The claims are therefore scoped to the analysis and computations available: verification of the original higher-order results and timing experiments would require the corresponding computational records. Taken together, the analysis quantifies how constitutive contrast affects stability and approximation, while the computational caveat defines the evidence available for independently assessing the reported numerical results.

Keywords
Error Estimates;
Finite Elements;
Newton Method;
Nonlinear Diffusion;
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1. INTRODUCTION

Nonlinear diffusion problems appear in non-Newtonian flow, nonlinear heat conduction and filtration. The abstract theory of strongly monotone Lipschitz operators goes back to Zarantonello (1960) and is presented in (Zeidler, 1990); finite element error estimates for the nonlinear equation $\nabla\cdot(a(x,u)\nabla u)=f$, whose coefficient depends on u rather than on ∇u, were obtained in (Douglas and Dupont, 1975), and the genuinely degenerate/singular case $a(\xi)=|\xi|^{p-2}\xi$, $p\neq 2$, requires weighted or quasi-norm arguments (Barrett and Liu, 1993; Ebmeyer and Liu, 2005; Diening and Kreuzer, 2008). General finite element background can be found in (Ciarlet, 1978/2002; Brenner and Scott, 2008; Ern and Guermond, 2004).

The purpose is a unified analysis for uniformly monotone radial fluxes, with explicit constitutive constants and a precise distinction between global damped Newton convergence and local convergence from nested finite element solutions. Quasi-optimality and iterative linearisation are established results (Congreve and Wihler, 2017; Heid and Wihler, 2020a, 2020b). The nested-start estimate is an elementary specialization of the mesh-independence principle (Allgower et al., 1986; Deuflhard and Potra, 1992), not a new general principle. The contribution is the combined derivation, computable law-dependent bounds, and numerical comparison. Section 4 states the additional dual-regularity assumption needed for its L^2 estimate. Reported experiments and independently recomputed checks are distinguished in Section 6.

2. CONTINUOUS PROBLEM

Let $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ be a bounded polygonal/polyhedral Lipschitz domain, $V=H_0^1(\Omega)$ with $\|v\|:=\|\nabla v\|_{L^2(\Omega)}$ (a Hilbert norm by Poincaré's inequality), and $f\in V^*$. Let $a(\xi)=\mu(|\xi|)\xi$ and consider the boundary value problem

$$-\nabla\cdot a(\nabla u)=f \text{ in } \Omega,\quad u|_{\partial\Omega}=0. \qquad (1)$$

Assumption 1. $\mu\in C^1([0,\infty))$, $\mu'(0)=0$, and there are constants $0<m_0\le M_0<\infty$ with

$$m_0\le\big(t\mu(t)\big)'=\mu(t)+t\mu'(t)\le M_0\quad\forall t\ge 0. \qquad (2)$$

Lemma 2.1. Under Assumption 1, $a\in C^1(\mathbb{R}^d;\mathbb{R}^d)$ with

$$Da(\xi)=\mu(|\xi|)I+\frac{\mu'(|\xi|)}{|\xi|}\xi\xi^{\mathsf{T}}, \qquad (3)$$

which is symmetric with spectrum in $[m_0,M_0]$, and for all $\xi,\eta\in\mathbb{R}^d$

$$\big(a(\xi)-a(\eta)\big)\cdot(\xi-\eta)\ge m_0|\xi-\eta|^2. \qquad (4a)$$

$$|a(\xi)-a(\eta)|\le M_0|\xi-\eta|. \qquad (4b)$$

Proof. For $\xi\neq 0$, $Da(\xi)$ has the eigenvalue $\mu(|\xi|)$ on $\xi^{\perp}$ and $\mu+t\mu'$ $(t=|\xi|)$ on $\operatorname{span}\xi$. The second lies in $[m_0,M_0]$ by (2); the first equals $\frac{1}{t}\int_0^t\big(s\mu(s)\big)'\,ds$, an average of values in $[m_0,M_0]$. Since $|\mu'(t)|t\to 0$ as $t\to 0$, Da extends continuously by $Da(0)=\mu(0)I$. Writing $a(\xi)-a(\eta)=\int_0^1 Da\big(\eta+s(\xi-\eta)\big)(\xi-\eta)\,ds$ and using the spectral bounds gives Eqs. (4a)–(4b).

▫

Define $A:V\to V^*$ by $\langle A(u),v\rangle=\int_\Omega a(\nabla u)\cdot\nabla v\,dx$. By Lemma 2.1, for all $u,v,w\in V$

$$\langle A(u)-A(v),u-v\rangle\ge m_0\,\|u-v\|^2. \qquad (5a)$$

$$\|A(u)-A(v)\|_{V^*}\le M_0\,\|u-v\|. \qquad (5b)$$

The weak form of (1) is: find $u\in V$ with

$$\langle A(u),v\rangle=\langle f,v\rangle\quad\forall v\in V. \qquad (6)$$

Let $\Phi(t)=\int_0^t s\,\mu(s)\,ds$; then A is the derivative of the strictly convex functional

$$E(v)=\int_\Omega\Phi\big(|\nabla v|\big)\,dx-\langle f,v\rangle. \qquad (7)$$

Example 2.2 (Concrete laws). (a) Shear-thinning (Carreau-type, $p=3/2$): $\mu(t)=1+(1+t^2)^{-1/4}$. Then $(t\mu)'=1+(1+t^2)^{-5/4}(1+t^2/2)$, and $g(s)=(1+s/2)(1+s)^{-5/4}$ $(s=t^2)$ satisfies $g'(s)=-(1+s)^{-9/4}\left(\frac{3}{4}+\frac{s}{8}\right)<0$, $g(0)=1$, $g(\infty)=0$. Hence $m_0=1$, $M_0=2$. (b) Saturating shear-thickening: $\mu(t)=1+\frac{t^2}{1+t^2}$. Then $(t\mu)'=1+\frac{3s+s^2}{(1+s)^2}$, whose second term vanishes at $s=0$, tends to 1, and has maximum $9/8$ at $s=3$. Hence $m_0=1$, $M_0=17/8$. (c) Strongly shear-thinning: $\mu(t)=\mu_\infty+(1-\mu_\infty)(1+t^2)^{-1/4}$ with $\mu_\infty=0.01$. Then $(t\mu)'=\mu_\infty+\big(1-\mu_\infty\big)g(t^2)$ with g from (a), so $(t\mu)'\in(\mu_\infty,1]$: $m_0=0.01$, $M_0=1$, $M_0/m_0=100$ (a zero-to-infinite-shear viscosity ratio of two orders of magnitude, as used for polymer-type fluids). All three laws satisfy $\mu'(0)=0$ and Da is globally Lipschitz.

Remark 2.3 (Constants). The values m_0,M_0 and the upper bounds for the Lipschitz constant of Da are explicit. Interpolation, regularity and patch-overlap constants remain generic. Inverse constants are explicit below only for piecewise linear elements on the stated uniform mesh.

Remark 2.4 (The p-Laplacian is excluded). For $a(\xi)=|\xi|^{p-2}\xi$ one has $(t\mu)'=(p-1)t^{p-2}$, which vanishes at $t=0$ and is unbounded as $t\to\infty$ if $p>2$,

and is unbounded at $t = 0$ and tends to 0 as $t \to \infty$ if $p < 2$; in neither case do uniform bounds m_0, M_0 exist. A condition of type $\langle A(u) - A(v), u - v \rangle \geq \alpha \| u - v \|^p$ together with $\| A(u) - A(v) \| \leq L \| u - v \|^{p-1}$ is therefore not satisfied by the p-Laplacian unless $p = 2$: for $p > 2$ take $u = tw + \delta z$, $v = tw$ with $t \to \infty$ (the difference behaves like $t^{p-2}\delta \gg \delta^{p-1}$); for $p < 2$ the first inequality fails by the same scaling. A regularised law $(\varepsilon^2 + t^2)^{(p-2)/2}$ satisfies (2) only after a further truncation or the addition of a constant ($\mu_\infty > 0$ for $p < 2$), as in Example 2.2(a). Adding a constant alone does not bound the growing derivative for $p > 2$.

Theorem 2.5 (Well-posedness and stability). Under Assumption 1, for each $f \in V^*$ there is a unique $u \in V$ with $A(u) = f$, and $\| u \| \leq m_0^{-1} \| f \|_{V^*}$. The same holds on every closed subspace $V_h \subset V$ for the Galerkin problem, with the same constants.

Proof. Let $R: V^* \to V$ be the inverse Riesz isomorphism and $T(v) = v - \rho R(A(v) - f)$. For $0 < \rho < 2m_0/M_0^2$, expanding the squared norm and using Eq. (5) gives $\| T(v) - T(w) \|^2 \leq q^2 \| v - w \|^2$, where $q^2 = 1 - 2\rho m_0 + \rho^2 M_0^2 < 1$. The contraction theorem supplies the unique solution. Testing the equation with u and using $a(0) = 0$ yields the stability bound. The same argument uses the restricted inverse Riesz map on any closed subspace. ▫

Proposition 2.6 (Lipschitz dependence on the data). If $u_i \in V_h$ (or V) solve $A(u_i) = f_i$ on the same space, then $\| u_1 - u_2 \| \leq m_0^{-1} \| f_1 - f_2 \|_{V_h^*}$. Hence the discrete solution map is Lipschitz continuous with a mesh-independent constant, which also yields the bound $\| u_h \| \leq \| f \|_{V^*}/m_0$.

Proof. $m_0 \| u_1 - u_2 \|^2 \leq \langle A(u_1) - A(u_2), u_1 - u_2 \rangle = \langle f_1 - f_2, u_1 - u_2 \rangle$.

▫

Proposition 2.7 (Zarantonello iteration). For $0 < \rho < 2m_0/M_0^2$ the iteration

$$u^{k+1} = u^k - \rho\, \mathcal{R}(A(u^k) - f) \quad (8)$$

on V or on V_h ($\mathcal{R}$ the Riesz map, i.e. the inverse of the stiffness matrix) satisfies $\| u^{k+1} - u \| \leq q(\rho) \| u^k - u \|$ with $q(\rho)^2 = 1 - 2\rho m_0 + \rho^2 M_0^2$; the best choice $\rho = m_0/M_0^2$ gives $q = \sqrt{1 - (m_0/M_0)^2}$.

Proof. This is the contraction estimate in the proof of Theorem 2.5.

▫

Proposition 2.8 (Conditioning of the Jacobian). Let $J(U)$ be the Jacobian of the discrete problem at $u_h \in V_h$ and K the stiffness matrix of the Laplacian on V_h (nodal basis, homogeneous Dirichlet conditions). Then $m_0\, W^\top K W \leq W^\top J(U) W \leq M_0\, W^\top K W$ for all W; hence all generalised eigenvalues of $J(U)v = \lambda K v$ lie in $[m_0, M_0]$ and $\kappa(J(U)) \leq (M_0/m_0)\, \kappa(K)$.

Proof. $W^\top J(U) W = \int_\Omega D\, a(\nabla u_h) \nabla w_h \cdot \nabla w_h$, and the spectrum of Da lies in $[m_0, M_0]$ (Lemma 2.1). The condition-number bound follows from $\lambda_{\max}(J) \leq M_0 \lambda_{\max}(K)$ and $\lambda_{\min}(J) \geq m_0 \lambda_{\min}(K)$.

▫

Proposition 2.9 (Kačanov iteration). Assume additionally that μ is nonincreasing. Given $u^n \in V_h$, define u^{n+1} by

$$\int_\Omega \mu\,(|\nabla u^n|) \nabla u^{n+1} \cdot \nabla v = \langle f, v \rangle. \quad (9)$$

This holds for every $v \in V_h$. Put $\delta_n = u^{n+1} - u^n$, $D_n = E(u^n) - E(u_h)$ and $q_E = 1 - (m_0/M_0)^2$. Then

$$D_n - D_{n+1} \geq 1/2\, m_0 \| \delta_n \|^2, \quad (9a)$$

and $D_{n+1} \leq q_E D_n$. Consequently $\| u^n - u_h \|^2 \leq 2 q_E^n D_0/m_0$.

Proof. Since $\psi(s) = \Phi(\sqrt{s})$ has nonincreasing derivative $\psi'(s) = \mu(\sqrt{s})/2$, its tangent defines a quadratic majorant Q_n of E, equal to E at u^n. Equation (9) minimizes Q_n and implies Eq. (9a). The restricted residual norm $r_n = \| A(u^n) - f \|_{V_h^*}$ satisfies $r_n \leq M_0 \| \delta_n \|$. Strong convexity gives $D_n \leq r_n^2/(2m_0)$. Combining these three inequalities yields the energy contraction. The final estimate follows from $m_0 \| u^n - u_h \|^2/2 \leq D_n$. ▫

This is a special case of the abstract theory in (Heid and Wihler, 2020a; Heid and Wihler, 2020b); it requires μ nonincreasing (shear-thinning) and is not available for the shear-thickening law of Example 2.2(b).

3. FINITE ELEMENT DISCRETISATION AND QUASI-OPTIMALITY

Let $\{\mathcal{T}_h\}$ be shape-regular triangulations and $V_h = \{v \in V \cap C(\bar{\Omega}) : v|_K \in \mathbb{P}_k\}$. The discrete problem is: find $u_h \in V_h$ with

$$\langle A(u_h), v_h \rangle = \langle f, v_h \rangle \quad \forall v_h \in V_h; \quad (10)$$

it is uniquely solvable and satisfies $\| u_h \| \leq m_0^{-1} \| f \|_{V^*}$ independently of h (Theorem 2.5). Subtracting (10) from (6) gives the nonlinear Galerkin orthogonality

$$\langle A(u) - A(u_h), v_h \rangle = 0 \quad \forall v_h \in V_h. \quad (11)$$

Theorem 3.1 (Quasi-optimality). Under Assumption 1,

$$\| u - u_h \| \leq \sqrt{M_0/m_0} \inf_{v_h \in V_h} \| u - v_h \|. \quad (12)$$

$$\| u - u_h \| \leq (M_0/m_0) \inf_{v_h \in V_h} \| u - v_h \|. \quad (13)$$

If moreover $u \in H^{k+1}(\Omega)$ then $\| u - u_h \| \leq C\sqrt{M_0/m_0}\, h^k |u|_{H^{k+1}}$ with C depending only on k and shape regularity.

Proof. Put $e = u - u_h$. Strong monotonicity, Galerkin orthogonality and the Lipschitz bound give, for any $v_h \in V_h$, $m_0 \| e \|^2 \leq M_0 \| e \| \| u - v_h \|$. This proves Eq. (13).

For Eq. (12), consider $g(t) = E(u + tz)$ for fixed $z \in V$. Differentiation along this scalar path is justified by the bounded matrix Da and gives $m_0 \| z \|^2 \leq g''(t) \leq M_0 \| z \|^2$. Since $g'(0) = 0$, the energy gap $D(v) = E(v) - E(u)$ satisfies

$$1/2\, m_0 \| v - u \|^2 \leq D(v) \leq 1/2\, M_0 \| v - u \|^2.$$

The discrete solution minimizes E on V_h, so $D(u_h) \leq D(v_h)$. Combining the bounds and taking the infimum proves Eq. (12). Standard interpolation gives the rate (Ciarlet, 1978/2002; Brenner and Scott, 2008). This argument uses directional second derivatives along lines and does not require a globally Lipschitz Hessian on the infinite-dimensional space. ▫

The bound (13) holds for every strongly monotone Lipschitz operator, variational or not; the sharper (12) uses that A is the derivative of a convex functional. For the laws of Example 2.2 the constants in (12) are $\sqrt{2} \approx 1.414$, $\sqrt{17/8} \approx 1.458$ and 10. We do not claim that these constants are attained or sharp: for smooth solutions the computed ratio $\| u - u_h \| / \| u - I_h u \|$ is close to 1 (Section 6).

Theorem 3.2 (Residual reliability). Let $f \in L^2(\Omega)$ and let $\mathcal{T}_h$ be shape regular. For $K \in \mathcal{T}_h$ put $R_K = f + \nabla \cdot a(\nabla u_h)|_K$, and for interior faces E (edges in two dimensions) put $J_E = [[a(\nabla u_h) \cdot n]]$. With

$$\eta_h^2 = \sum_K h_K^2 \| R_K \|_{L^2(K)}^2 + \sum_E h_E \| J_E \|_{L^2(E)}^2 \quad (14)$$

one has

$$\| u - u_h \| \leq \frac{C_{rel}}{m_0} \eta_h, \quad (15)$$

where C_{rel} depends only on k and the shape regularity.

Proof. Define $\mathcal{R}(v) = \langle f, v \rangle - \langle A(u_h), v \rangle$. Strong monotonicity implies $m_0 \| u - u_h \| \leq \| \mathcal{R} \|_{V^*}$. Galerkin orthogonality gives $\mathcal{R}(v) = \mathcal{R}(v - I_C v)$ for the boundary-preserving Clément interpolant (Clément, 1975). Elementwise integration by parts expresses this residual using R_K and the flux jumps J_E. The local estimates $\| v - I_C v \|_{L^2(K)} \leq C h_K \| \nabla v \|_{L^2(\omega_K)}$ and $\| v - I_C v \|_{L^2(E)} \leq C h_E^{1/2} \| \nabla v \|_{L^2(\omega_E)}$, followed by Cauchy–Schwarz and finite patch overlap, yield $\| \mathcal{R} \|_{V^*} \leq C_{rel} \eta_h$ (Verfürth, 2013). ▫

Efficiency (a lower bound $\eta_h \lesssim \| u - u_h \|$ +data oscillation) is not proved here; Section 6 reports observed ratios. The factor $1/m_0$ makes the bound pessimistic for strongly nonlinear laws.

4. L^2 ESTIMATE

Assumption 2. Da is Lipschitz (any dimension d): $|Da(\xi) - Da(\eta)| \leq K|\xi - \eta|$ in the operator norm; this holds for all three laws of Example 2.2, with the constants of Remark 5.2.

Assumption 3. $d = 2$, Ω is a convex polygon, the meshes are quasi-uniform, $k \geq 2$, and $u \in W^{k+1,4}(\Omega)$. (By the Sobolev embedding, $W^{k+1,4}(\Omega) \subset H^{k+1}(\Omega) \cap W^{2,\infty}(\Omega)$ for $d = 2$, $k \geq 2$, so no additional primal regularity is needed.)

We also assume the following dual regularity for the linearisation at the exact solution: for every $g \in L^2(\Omega)$, the weak solution $z \in V$ of $B_u(v, z) = (g, v)$ satisfies $\| z \|_{H^2} \leq C_{reg} \| g \|_{L^2}$, with B_u defined below. This is a condition on the domain and the linearised coefficient, not on the mesh; standard elliptic regularity provides sufficient settings (Grisvard, 1985/2011).

Theorem 4.1 (L^2 estimate). Under Assumptions 1, 2 and 3 and the stated dual regularity,

$$\| u - u_h \|_{L^2(\Omega)} \leq C\, h^{k+1}, \quad (16)$$

with C depending on m_0, M_0, K, Ω, shape regularity and the norms of u.

Proof. Set $e = u - u_h$ and $B_u(w, v) = \int_\Omega D\, a(\nabla u) \nabla w \cdot \nabla v$. Let z solve the stated dual problem with $g = e$, and choose a conforming interpolant z_h satisfying $\| z - z_h \| \leq Ch \| z \|_{H^2}$ and $\| z_h \| \leq C \| z \|_{H^2}$. The dual-regularity bound and Theorem 3.1 imply

$$|B_u(e, z - z_h)| \leq Ch^{k+1} \| e \|_{L^2}.$$

Let $r = Da(\nabla u)\nabla e - a(\nabla u) + a(\nabla u_h)$. The Lipschitz derivative gives $|r| \leq K|\nabla e|^2/2$. Galerkin orthogonality yields $B_u(e, z_h) = \int_\Omega r \cdot \nabla z_h$, and hence

$$|B_u(e, z_h)| \leq C \| \nabla e \|_{L^4}^2 \| e \|_{L^2}.$$

Split $e = (u - I_h u) + (I_h u - u_h)$. The first gradient has L^4 norm at most Ch^k by the assumed regularity. For the second, the two-dimensional inverse estimate costs $h^{-1/2}$, while its energy norm is $O(h^k)$ by interpolation and Theorem 3.1. Therefore $\| \nabla e \|_{L^4} \leq Ch^{k-1/2}$. Since $\| e \|_{L^2}^2 = B_u(e, z)$, division by $\| e \|_{L^2}$ gives

$$\| e \|_{L^2} \leq C(h^{k+1} + h^{2k-1}) \leq Ch^{k+1}$$

for $k \geq 2$; the zero-error case is immediate. ▫

Remark 4.2. For $k = 1$ the term h^{2k-1} equals h, so the argument only yields $O(h)$: the nonlinear remainder $\int r \cdot \nabla I_h z$ with $|r| \lesssim |\nabla e|^2$ is controlled through $\| \nabla e \|_{L^4}^2 \sim h^{2k-1}$, and for $\mathbb{P}_1$ this is the order that the inverse estimate allows. One possible route to improving the estimate uses $\| \nabla e \|_{L^\infty} = O(h\,|\log h|^c)$, i.e. $W^{1,\infty}$ error estimates for the nonlinear Galerkin problem, which are known for linear problems on convex polygons but are not established here for the nonlinear problem. The experiments of Section 6 show order 2 also for $k = 1$ (also in three dimensions); this is observed, not proved here. In three dimensions the same inverse-estimate calculation requires $k \geq 3$, together with the corresponding primal and dual regularity; convexity of a polyhedron alone is not asserted to supply it.

5. NEWTON'S METHOD

In coordinates of a nodal basis, $F(U) = 0$ is the discrete system and $J(U)$ its Jacobian, symmetric positive definite: $\langle J(U)W, W\rangle = \int_\Omega D\, a(\nabla u_h)\nabla w_h \cdot \nabla w_h \in [m_0, M_0] \| w_h \|^2$. F is the gradient of $E_h = E|_{V_h}$, so Newton's method for $F = 0$ is Newton's method for minimising a strongly convex function. We measure everything in the energy norm $\|\cdot\|$ and its dual (Newton's method is affine invariant), and use the Newton iteration

$$J(U_k)\delta_k = -F(U_k), \quad U_{k+1} = U_k + t_k\delta_k, \quad (17)$$

the Newton decrement

$$\lambda_k^2 = F(U_k)^\top J(U_k)^{-1} F(U_k), \quad (18)$$

and Armijo backtracking, i.e. $t_k = \beta^j$ with the smallest $j \geq 0$ such that

$$E_h(U_k + t_k\delta_k) \leq E_h(U_k) - \sigma t_k \lambda_k^2. \quad (19)$$

Lemma 5.1. Assume the quasi-uniform, shape-regular mesh family of fixed polynomial degree, and Assumption 2. Let $C_{4,inv}$ satisfy $\| \nabla w_h \|_{L^4} \leq C_{4,inv} h^{-d/4} \| w_h \|$. With

$$L_h = K C_{4,inv}^2 h^{-d/2},$$

the Hessian $H_h(v) = E_h''(v)$ satisfies $\| H_h(v) - H_h(w) \| \leq L_h \| v - w \|$. Here the operator norm is from V_h to V_h^*.

Proof. Test the Hessian difference on $\phi, \psi \in V_h$. Assumption 2 bounds its integrand by $K|\nabla(v - w)||\nabla\phi||\nabla\psi|$. Hölder with exponents $(4,4,2)$ and two inverse estimates give the stated operator bound. ▫

Remark 5.2 (Explicit K and radius). Write $Da(\xi) = \mu(t)I + q(t^2)\xi\xi^\top$ with $t = |\xi|$, $q(s) = 2\,d\mu/ds$. For a unit vector v,

Define $A_v = \mu'(t)(\xi \cdot v)I/t$, $B_v = 2q'(t^2)(\xi \cdot v)\xi\xi^\top$ and $C_v = q(t^2)(v\xi^\top + \xi v^\top)$. The continuous value is used at $t = 0$. Then

$$\partial_v Da(\xi) = A_v + B_v + C_v, \quad (20)$$

so

$$K \leq \sup_{t \geq 0} (|\mu'(t)| + 2|q(t^2)|t + 2|q'(t^2)|t^3). \quad (21)$$

Elementary calculus (locating the maxima of $t(1 + t^2)^{-5/4}$, $t^3(1 + t^2)^{-9/4}$ and of the analogous functions for the thickening law) gives $K \leq 0.946$ (Carreau-type), $K \leq 2.949$ (thickening), $K \leq 0.936$ (strongly thinning); a grid search over $|\xi| \leq 60$ and all directions gives the sharper numerical values 0.479, 1.457, 0.474 (numerical search reported in the source). For $\mathbb{P}_1$ on the uniform mesh of the unit square, $\| \nabla w_h \|_{L^\infty} \leq \sqrt{2}\, h^{-1} \| \nabla w_h \|_{L^2}$ (a single element carries $\| \nabla w_h \|_{L^\infty}^2 |T| \leq \| \nabla w_h \|^2$ with $|T| = h^2/2$), so the L^∞ inverse constant $C_{inv} \leq \sqrt{2}$, and the radius $R = 2m_0/(KC_{inv})$ of Theorem 5.5 is at least 1.49, 0.47 and 0.015 for the three laws. For higher degrees C_{inv} depends on k and the mesh and is not computed. These radii only quantify the local quadratic estimate (23): the mesh-independence conclusion of Theorem 5.5 needs $k > d/2$ and therefore does not apply to $\mathbb{P}_1$ in two dimensions (Remark 5.6).

Theorem 5.3 (Damped Newton). Under the mesh assumptions of Lemma 5.1 and Assumptions 1–2, Armijo-damped Newton converges from every starting vector, for $0 < \sigma < 1/2$ and $0 < \beta < 1$. For $L_h > 0$, put $c_\sigma = \min\{1, 3(1 - 2\sigma)\}$ and

$$\eta = c_\sigma m_0^2 / L_h, \qquad \gamma = \sigma\beta\eta^2 m_0 / M_0^2.$$

Write $r_n = \| F(U_n) \|_{V_h^*}$ and $D_0 = E_h(U_0) - E_h(U^*)$. At most D_0/γ steps have $r_n \geq \eta$. Once $r_n < \eta$, the full step is accepted and

$$r_{n+1} \leq \frac{L_h}{2m_0^2} r_n^2.$$

This region is invariant and convergence is quadratic.

Proof. Let $r = \| F(U) \|_{V_h^*}$ and $\delta = -J(U)^{-1}F(U)$. Then $\| \delta \| \leq r/m_0$ and $\lambda^2 \geq r^2/M_0$. The Hessian bound gives

$$E_h(U + t\delta) \leq E_h(U) - t\lambda^2 + 1/2\, M_0 t^2 \| \delta \|^2. \quad (21a)$$

Consequently every $t \leq 2(1 - \sigma)m_0/M_0$ passes Armijo, and backtracking accepts $t \geq \beta m_0/M_0$. Thus each step with $r \geq \eta$ decreases energy by at least γ. For a full step, the Hessian-Lipschitz remainder is at most $L_h \| \delta \|^3/6$. If $r < \eta$, this is bounded by $(1/2 - \sigma)\lambda^2$, proving acceptance. Taylor expansion of the residual gives

$$\| F(U + \delta) \|_{V_h^*} \leq \frac{L_h}{2m_0^2} r^2. \quad (21b)$$

Because $\eta \leq m_0^2/L_h$, this region is invariant and the stated quadratic estimate follows. The finite energy

gap permits only finitely many steps outside it; strong monotonicity then implies convergence to U^*. When $L_h = 0$, the discrete problem is affine and one full Newton step solves it. ▫

Remark 5.4 (General mesh dependence). m_0 and M_0 do not depend on h, but $L_h \sim h^{-d/2}$, so the worst-case radius η of the quadratic phase shrinks with h and the bound in (i) grows. This global estimate alone does not give mesh-independent iteration counts. Theorem 5.5 addresses a more restrictive nested-start regime.

Theorem 5.5 (Newton from nested starting values). Let $d = 2,3$, let the meshes be quasi-uniform, assume that Da is Lipschitz with constant K and assume $K > 0$, and let C_{inv} be the constant in $\| \nabla w_h \|_{L^\infty} \le C_{inv} h^{-d/2} \| \nabla w_h \|_{L^2}$ for $w_h \in V_h$. Put

$$R = \frac{2m_0}{K\, C_{inv}}. \quad (22)$$

Let $U_0 \in V_h$, $e_n = U_n - u_h$ for the undamped Newton iterates. Then

$$\| e_{n+1} \| \le \frac{1}{R\, h^{d/2}} \| e_n \|^2. \quad (23)$$

In particular, if $\| e_0 \| \le \frac{1}{2} R h^{d/2}$, then $\| e_n \| \le R h^{d/2}\, 2^{-2^n}$ for all $n \ge 0$. If moreover both $\| U_0 - u \| \le C_0 h^s$ and $\| u - u_h \| \le C_1 h^s$ hold with $s > d/2$ (for instance $U_0 = u_{2h}$, the Galerkin solution on a coarser nested quasi-uniform mesh with $V_{2h} \subset V_h$, with $s = k$ by Theorem 3.1 when $u \in H^{k+1}$ and $k > d/2$), then there is $h_0 > 0$ such that for all $h \le \min(h_0, 1)$ the number of Newton steps needed to reach $\| e_n \| \le \tau$ is at most $N_\tau = \lceil \log_2 \left(\max\{1, \log_2(R/\tau)\} \right) \rceil$, independently of h.

Proof. For $e_n = U_n - u_h$, Newton's equation and the integral form of the derivative give $J(U_n) e_{n+1} = \int_0^1 [J(U_n) - J(u_h + s e_n)]\, e_n\, ds$. Testing against v_h bounds the integrand by

$$K(1 - s) \| \nabla e_n \|_{L^\infty} \| e_n \| \| v_h \|.$$

The inverse estimate and integration in s supply the factor $K C_{inv} h^{-d/2}/2$. Coercivity of $J(U_n)$ then gives Eq. (23). With $q_n = \| e_n \| / \left(R h^{d/2} \right)$, this is $q_{n+1} \le q_n^2$. Thus $q_0 \le 1/2$ implies the double-exponential bound. The two approximation assumptions give $\| e_0 \| \le (C_0 + C_1) h^s$, so $q_0 \to 0$ when $s > d/2$. For sufficiently small $h \le 1$, $\| e_n \| \le R 2^{-2^n}$, proving the stated nonnegative bound for every $\tau > 0$. ▫

Remark 5.6. Theorem 5.5 is an instance of the mesh-independence principle for Newton's method, in an elementary form for the present class with explicit radius; it holds only for starting values whose energy error is $o\left(h^{d/2}\right)$, i.e. it does not contradict the $h^{-d/2}$-dependence of the general result Theorem 5.3. It does not cover $k = 1$ with $d = 2$, where $s = d/2$ and q_0 is only bounded by a constant.

6. NUMERICAL RESULTS AND DISCUSSION

The reported computations use $u(x, y) = \sin(\pi x) \sin(\pi y)$ on the unit square, uniform diagonal triangulations with $h = 1/N$, and continuous $\mathbb{P}_k$ elements. The forcing is obtained analytically from the stated flux. The original document reports a 7×7 collapsed Gauss rule, sparse direct solves, Armijo halving with $\sigma = 10^{-4}$, and termination at Newton decrement 10^{-11}. Its code and complete logs were not attached. Tables 2–4 and Figures 1–2 therefore retain reported results, except that the $\mathbb{P}_1$ errors were independently recomputed for all three laws at $N = 8,16,32,64$ with a separate implementation supplied with this revision. These checks reproduce the displayed $N = 64$ errors and the four-to-five-step Newton behavior; they do not validate the higher-degree, timing or extended experiments.

Table 1 collects the constitutive bounds. The upper bounds for K follow from separate maxima of the three terms in Eq. (21): for the first law they sum to 0.9452390493, for thickening to 2.9485571586, and for strong thinning to 0.99 times the first value. The listed upper bounds are rounded upwards and the radii downwards. The numerical suprema are estimates retained from the original search, not certified global maxima. The $\mathbb{P}_1$ radii quantify local convergence only; $k > d/2$ fails for $k = 1, d = 2$.

Table 1. Constitutive constants and local radius. C: Carreau-type; T: thickening; S: strongly thinning. Radius bounds are for P₁ only.

Law	m_0	M_0	K upper	R lower
C	1	2	.946	1.49
T	1	2.125	2.949	.47
S	.01	1	.936	.015

The energy errors in Table 2 follow the expected order k. The reported L^2 orders $k + 1$ agree with Theorem 4.1 for $k \ge 2$ under its assumptions; the order two for $k = 1$ remains an observation. Ratios relative to the interpolant range from approximately 0.85 to 1.00, below $\sqrt{M_0/m_0}$, but do not establish sharpness of that bound. Figure 1 is retained from the supplied manuscript and displays the reported refinement series.

Table 2. Errors at h = 1/64 and rates from h = 1/32. E_1 denotes the gradient L^2 error and E_0 the L^2 solution error.

Law	k	E_1	Rate	E_0	Rate
C	1	5.45e−2	1.00	3.57e−4	2.00
C	2	5.28e−4	2.00	1.08e−6	3.00
C	3	3.21e−6	3.00	4.66e−9	4.01
T	1	5.45e−2	1.00	3.23e−4	2.00
T	2	5.28e−4	2.00	1.08e−6	3.00
T	3	3.21e−6	3.00	4.66e−9	4.01
S	1	5.45e−2	1.00	4.07e−4	2.00
S	2	5.28e−4	2.00	1.08e−6	3.00
S	3	3.21e−6	3.00	4.66e−9	4.01

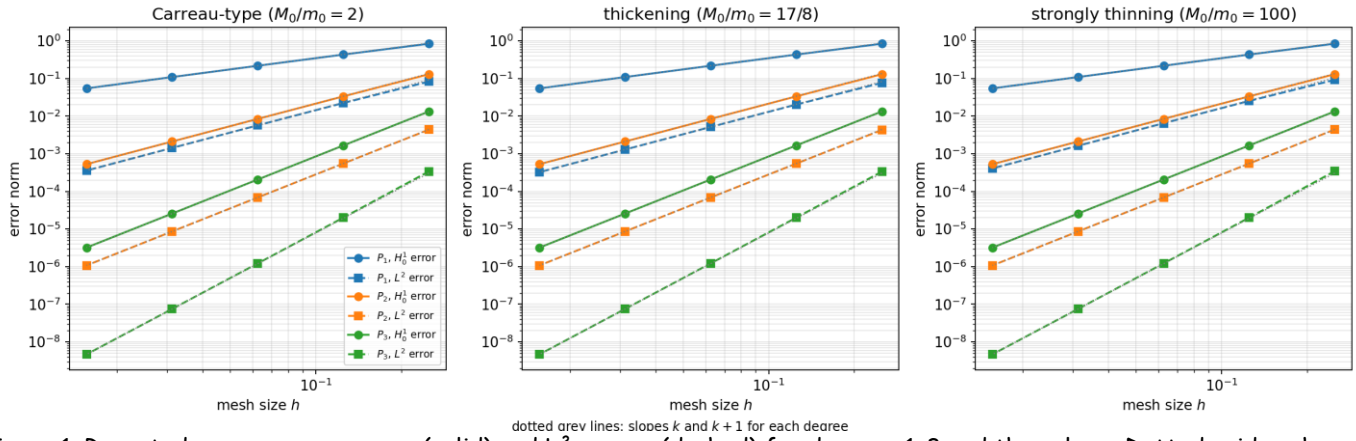


Figure 1. Reported energy-norm errors (solid) and L^2 errors (dashed) for degrees 1–3 and three laws. Dotted guides show slopes k and k + 1. Original supplied figure; only the P_1 results were independently recomputed.

Newton required four or five steps from zero in the baseline experiments. Table 3 compares it with three fixed-point schemes. The weak Zarantonello bound for strong thinning is $\sqrt{1-10^{-4}} = 0.9999499987$, not exactly one. Picard convergence is proved here only for nonincreasing viscosity. Its reported convergence for thickening is an additional observation. Overlap of timing standard deviations does not establish statistical equivalence, and no universal timing advantage is claimed.

Table 3. Reported iterations (error-contraction factor), P_2, h = 1/32. A cap is not convergence.

Law	Newton	Picard	Chord	Zar.
C	4	14 (.16)	22 (.33)	59 (.66)
T	4	21 (.33)	>3000 (1.07)	68 (.77)
S	5	27 (.42)	51 (.65)	>400 (.996)

Nested-start counts decrease to two for the reported quadratic and cubic cases (Table 4). This is consistent with Theorem 5.5, but bounded counts for linear elements are outside that theorem. Figure 2 shows reported decrement histories. The implementation's stated switch to full steps below decrement 10^{-5} is a numerical safeguard, not the exact Armijo algorithm in Theorem 5.3. The energy and residual criteria must be checked explicitly when reproducing those runs.

Table 4. Reported Newton steps from zero / nested start. The P_3 nested-start experiment at h = 1/64 was not reported.

Law	k	1/8	1/16	1/32	1/64
C	1	4/4	4/4	4/3	4/3
C	2	4/3	4/3	4/2	4/2
C	3	4/3	4/2	4/2	—
T	1	5/4	4/4	5/4	5/3
T	2	5/4	5/3	5/2	5/2
T	3	5/3	5/2	5/2	—
S	1	5/4	5/4	5/4	5/3
S	2	5/3	5/3	5/3	5/2
S	3	5/3	5/2	5/2	—

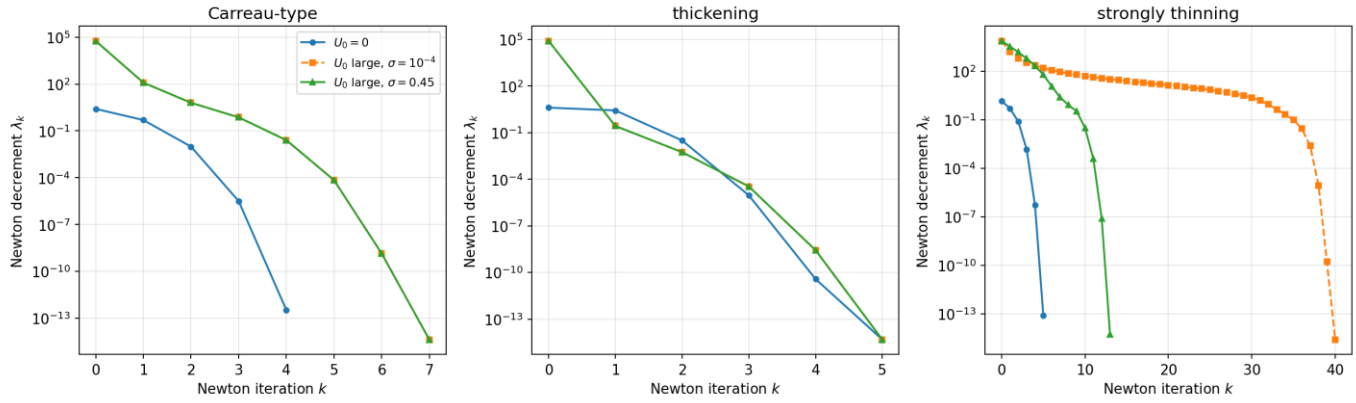


Figure 2. Reported Newton-decrement histories for P_2, h = 1/32, from zero and large starts, with default and demanding Armijo parameters. Original supplied figure; the large-start histories were not independently rerun.

The supplementary document preserves the additional reported tests: unstructured and three-dimensional meshes, an L-shaped domain, duct flow, regularised p-Laplacian comparisons, estimator behavior, large-start damping and solver costs. Their original input files and logs remain necessary for full verification. In particular, an extrapolated flow-rate reference is not a rigorous error bound, an L-shaped-domain energy diagnostic is not an exact norm error, and a generic ILU preconditioner does not ensure the symmetry and positivity required by conjugate gradients.

7. LIMITATIONS AND CONCLUSION

For uniformly monotone radial diffusion, the energy argument yields quasi-optimality with factor $\sqrt{M_0/m_0}$ and a residual-reliability estimate. A quadratic nonlinear remainder yields the stated L^2 order for degrees at least two in two dimensions, with explicit primal and dual regularity conditions. Damped Newton converges globally for each fixed discretisation, whereas the mesh-independent local iteration bound requires starting errors smaller than $h^{d/2}$; nested Galerkin starts meet this condition when $k > d/2$ and the solution is sufficiently regular.

The theory excludes untruncated degenerate or singular p-Laplacian laws. It does not prove estimator efficiency, adaptive optimality, inexact-Newton performance, mesh independence from zero, or the observed linear-element L^2 rate. The independent linear-element calculations support the core reported convergence values. Complete reproduction of the remaining experiments requires the original computational archive. The next analytical steps are sharper constitutive constants, weaker regularity assumptions and a controlled treatment of linearisation and discretisation errors within adaptive schemes.

DECLARATIONS

Data and code. The accompanying archive supplies an independent $\mathbb{P}_1$ verification script and its outputs, the original figures, and the extended numerical report. It does not contain the original higher-order solver or timing logs.

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