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Nonlocal Reduction of Order and Repeated Roots for Sequential Caputo Differential Equations

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Abstract

We study the multiplicative substitution for linear fractional differential equations containing two sequential Caputo derivatives of orders between zero and one. The analysis first identifies a limitation of finite endpoint data: a polynomial counterexample proves that no fixed finite collection of endpoint derivatives and initial values determines a Caputo derivative. This obstruction to finite-jet reconstruction does not prevent elimination of the additive constant in the multiplier. Under explicit absolute-continuity assumptions, an exact product-remainder identity transforms the original equation into a nonlocal equation for the first derivative of the transformed unknown. For constant coefficients and a repeated characteristic root, factorization of the operator yields a fundamental solution pair: a Mittag–Leffler eigenfunction and its derivative with respect to the spectral parameter. We verify the regularity required for the construction, establish the sequential initial traces and uniqueness, and prove compact-uniform convergence of the pair to the classical repeated-root solutions in the limiting regime. We identify this solution as a specialization of established matrix Mittag–Leffler theory, clarifying its relation to the fractional linear system framework. Recomputed numerical examples illustrate the formulas. Two experiments based on the L1 discretization examine residual behavior and provide computational checks on the derivation. These computations support the examples but do not replace the analytical arguments, establish a general convergence rate for nonsmooth data, or extend the repeated-root result beyond its stated constant-coefficient setting. The results therefore separate a local impossibility statement, an exact nonlocal transformation, and a rigorously characterized fundamental pair, while keeping numerical evidence within its demonstrated scope.

Keywords

Caputo derivative;
Mittag–Leffler function;
Nonlocal reduction;
Repeated roots;
Sequential equations

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1. INTRODUCTION

Reduction of order for a classical homogeneous second-order equation uses a known solution f and the substitution $y = fv$. The ordinary product rule cancels the term proportional to v and yields a first-order equation for v'. Fractional derivatives require more care because the derivative of a product generally depends on the history of both factors. Reduction methods developed for one fractional operator cannot be transferred solely by replacing the derivative symbol.

Abu Hammad et al. (2018) discuss reduction techniques for conformable equations, while Al-Refai (2018) treats a class involving the Caputo–Fabrizio derivative. These operators differ from the singular-kernel Caputo derivative considered here. Generalized Leibniz expansions, such as those studied by Osler (1970), can describe fractional products under appropriate hypotheses, but an infinite expansion alone is not a proof that all order-reducing constructions are impossible. In particular, a term containing $v(x) - v(0)$ can be written as an integral of v'.

This paper separates two questions: whether the ordinary finite local product rule remains valid, and whether the multiplicative substitution can still eliminate the additive constant in v. We give a polynomial test showing that the Caputo derivative cannot be recovered from any fixed finite collection of endpoint derivatives, even if the initial value is supplied. We then derive an exact integral identity that expresses the transformed equation in terms of $w = v'$. The resulting equation retains memory and is not an ordinary first-order differential equation.

For constant coefficients with a repeated characteristic root, the sequential operator factors into two first-order Caputo operators. The corresponding fundamental pair is obtained by differentiating a Mittag–Leffler eigenfunction with respect to the spectral parameter. This pair also follows from established matrix Mittag–Leffler theory (Garrappa and Popolizio, 2018). It is presented here as a transparent specialization and a test case for the nonlocal identity, rather than as a newly discovered special-function solution. The relation with Jornet's (2024) L-fractional construction is discussed with explicit attention to the different operators.

The analysis is accompanied by a proof of the classical limit, a worked example, and two L1 consistency experiments. No physical critical-damping threshold, general convergence rate for nonsmooth data, or exhaustive priority claim is inferred from these calculations.

2. DEFINITIONS AND ANALYTICAL SETTING

Fix $0 < \alpha < 1$. All derivatives have lower terminal zero. For an absolutely continuous function u on $[0, T]$, write

$$D_\alpha u(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha}\, u'(t)\, dt. \qquad (1)$$

The integral is defined almost everywhere. Whenever it has a continuous extension to zero, $D_\alpha u(0)$ denotes its right-hand limit, not a value obtained by substituting zero into the integral. This convention permits nonzero sequential initial data for profiles behaving like x^α (Caputo, 1967; Diethelm, 2010).

The sequential operator used throughout is $S_\alpha u = D_\alpha(D_\alpha u)$. It is not identified with the direct Caputo derivative of order 2α. Our solution class on a finite interval is

$$\mathcal{A}_\alpha[0,T] = \{u \in AC[0,T] : D_\alpha u \in AC[0,T]\}. \qquad (2)$$

Equations hold almost everywhere unless pointwise validity is stated. We consider continuous coefficients P, Q and

$$\mathcal{L}u = S_\alpha u + P(x) D_\alpha u + Q(x) u. \qquad (3)$$

For example, $u = x^\alpha/\Gamma(\alpha+1)$ has $D_\alpha u = 1$ and $S_\alpha u = 0$. At $\alpha = 1/2$, its ordinary first derivative is nonzero for $x > 0$. Thus even in that case the sequential operator is not the ordinary derivative.

The Mittag–Leffler function and its derivative with respect to the argument are

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \qquad E'_\alpha(z) = \frac{d}{dz} E_\alpha(z). \qquad (4)$$

The series are entire. With $F_\lambda(x) = E_\alpha(\lambda x^\alpha)$, termwise Caputo differentiation gives $D_\alpha F_\lambda = \lambda F_\lambda$. The standard Laplace pairs are $\mathcal{L}\{D_\alpha u\} = s^\alpha U - s^{\alpha-1} u(0)$ and $\mathcal{L}\{F_\lambda\} = s^{\alpha-1}/(s^\alpha - \lambda)$, in a right half-plane where the transforms exist (Podlubny, 1999).

3. LOCAL OBSTRUCTION AND NONLOCAL REDUCTION

3.1. Failure of a finite local differentiation rule

Proposition 1. Fix $x_* > 0$ and a nonnegative integer m. The value $D_\alpha v(x_*)$ on analytic functions is not determined by the data $v(0), v(x_*), v'(x_*), \dots, v^{(m)}(x_*)$.

Proof. Compare the zero function with $p(t) = t(x_* - t)^{m+1}$. The polynomial has $p(0) = 0$ and $p^{(j)}(x_*) = 0$ for $0 \le j \le m$. Integration by parts in (1) gives

$$D_\alpha p(x_*) = -\frac{\alpha}{\Gamma(1-\alpha)} \int_0^{x_*} \frac{p(t)}{(x_* - t)^{\alpha+1}}\, dt < 0. \qquad (5)$$

The boundary term vanishes because $p(0) = 0$ and $(x_* - t)^{-\alpha} p(t)$ tends to zero at x_*. The remaining integral is finite: its endpoint behavior is proportional to $(x_* - t)^{m-\alpha}$, whose exponent exceeds -1. The two functions therefore have identical prescribed data but different Caputo derivatives. This proves the statement. □

Taking the known factor $f = 1$ already rules out a universal product formula using only a fixed finite set of ordinary derivatives at the evaluation point and initial values. Proposition 1 does not exclude identities on restricted families of functions, integral transformations, or cancellations in a particular differential equation.

3.2. An exact product identity

For $f, v \in AC[0,T]$, define the product remainder

$$\mathcal{C}_f[v](x) = D_\alpha(fv)(x) - v(x) D_\alpha f(x). \qquad (6)$$

Direct substitution into (1), without any generalized-Leibniz series, yields

$$\mathcal{C}_f[v](x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} [f(t)v'(t) + f'(t)\big(v(t) - v(x)\big)]dt. \qquad (7)$$

This formula immediately shows that $\mathcal{C}_f[v + c] = \mathcal{C}_f[v]$ for every constant c. The apparent occurrence of an undifferentiated v is through differences, not through a dependence on its additive constant.

Put $w = v'$. Since $v(t) - v(x) = -\int_t^x w(s)\, ds$, Fubini's theorem rewrites (7) as $\mathcal{C}_f[v] = \mathcal{K}_f w$, where

$$\begin{aligned} \big(\mathcal{K}_f w\big)(x) &= \frac{1}{\Gamma(1-\alpha)} \int_0^x K_f(x,s) w(s)\, ds, \\ K_f(x,s) &= \frac{f(s)}{(x-s)^\alpha} - \int_0^s \frac{f'(t)}{(x-t)^\alpha}\, dt. \end{aligned} \qquad (8)$$

These identities hold almost everywhere. To justify the interchange, the absolute double integral in (7) is bounded, for almost every x, by $\| w \|_{L^1(0,T)} \int_0^x (x-t)^{-\alpha} |f'(t)| dt$. The latter convolution is finite almost everywhere. The first term is also finite almost everywhere because f is bounded and w is integrable.

3.3. Elimination of the multiplicative constant

Proposition 2. Let $f \in \mathcal{A}_\alpha[0,T]$ satisfy $\mathcal{L}f = 0$, and set $g = D_\alpha f$. Suppose $v \in AC[0,T]$ and $fv \in \mathcal{A}_\alpha[0,T]$. With $w = v'$, one has

$$\mathcal{L}(fv) = D_\alpha\big(\mathcal{K}_f w\big) + \mathcal{K}_g w + P\mathcal{K}_f w. \qquad (9)$$

Consequently, within this class, $y = fv$ solves $\mathcal{L}y = 0$ if and only if the right side of (9) vanishes.

Proof. Equation (6) gives $D_\alpha(fv) = vg + \mathcal{K}_f w$. Both $D_\alpha(fv)$ and vg are absolutely continuous, so $\mathcal{K}_f w$ is absolutely continuous and its Caputo derivative is defined. Applying (6) with factor g gives $D_\alpha(vg) = vD_\alpha g + \mathcal{K}_g w$. Substitution into (3) yields $v\mathcal{L}f$ plus the right side of (9). The former vanishes by hypothesis. □

The transformed equation depends on w and its history but not on the integration constant in $v(x) = c + \int_0^x w(s) ds$. Varying c adds the known solution cf. This is a nonlocal reduction in the multiplicative variable; it is not a claim that (9) is a local first-order ordinary equation or a general practical inversion formula. If f has no zeros on $[0,T]$, every $y \in \mathcal{A}_\alpha[0,T]$ has an absolutely continuous quotient $v = y/f$, and the representation applies to the entire solution class on that interval. No such global assertion is made across zeros of f.

For comparison, under hypotheses sufficient for a convergent Riemann–Liouville Leibniz expansion, the same remainder can be represented as

$$\mathcal{C}_f[v](x) = \sum_{k=1}^{\infty} \binom{\alpha}{k} v^{(k)}(x) I^{k-\alpha} f(x) + B_f[v](x), \qquad (10a)$$

$$B_f[v](x) = \frac{f(0) x^{-\alpha}}{\Gamma(1-\alpha)} [v(x) - v(0)]. \qquad (10b)$$

Here I^β is the Riemann–Liouville integral. Osler's (1970) product formulas motivate this representation; we do not assert convergence for every analytic factor or pass a second Caputo derivative through it. Equation (9) instead follows from the integral definition. Since the boundary difference in (10b) equals $\int_0^x w$, it cannot establish that elimination of the additive constant fails. What fails universally is a finite local differentiation rule, as Proposition 1 demonstrates.

4. THE REPEATED-ROOT EQUATION

4.1. Fundamental pair and sequential initial data

Let $b, c \in \mathbb{R}$, $b^2 = 4c$, and $\lambda = -b/2$. Then

$$S_\alpha y + bD_\alpha y + cy = (D_\alpha - \lambda)^2 y = 0. \qquad (11)$$

The square denotes composition, with constants acting by multiplication. Define

$$F(x) = E_\alpha(\lambda x^\alpha), \qquad H(x) = x^\alpha E'_\alpha(\lambda x^\alpha). \qquad (12)$$

Theorem 1. The functions F, H belong to $\mathcal{A}_\alpha[0,T]$ for every finite T, form a fundamental pair for (11), and satisfy

$$D_\alpha F = \lambda F, \qquad D_\alpha H = F + \lambda H. \qquad (13)$$

For prescribed $y(0) = y_0$ and $D_\alpha y(0) = g_0$, the unique solution in this class is

$$y(x) = y_0 F(x) + (g_0 - \lambda y_0) H(x). \qquad (14)$$

Proof. Expanding the second function gives

$$H(x) = \sum_{n=1}^{\infty} \frac{n\lambda^{n-1} x^{n\alpha}}{\Gamma(n\alpha + 1)}. \qquad (15)$$

For $\lambda = 0$ this means $H = x^\alpha/\Gamma(\alpha + 1)$. On $[0,T]$, the sum of the L^1 norms of the ordinary derivatives of the terms in (15) is

$$\sum_{n=1}^{\infty} \frac{n|\lambda|^{n-1} T^{n\alpha}}{\Gamma(n\alpha + 1)} < \infty. \qquad (16)$$

Thus the series defines an absolutely continuous function. The analogous argument applies to F. Integrating the absolute termwise derivatives against the Caputo kernel gives a convergent series for every $x > 0$, so termwise application of (1) is justified. Reindexing yields

$$D_\alpha H = \sum_{m=0}^{\infty} \frac{(m+1)\lambda^m x^{m\alpha}}{\Gamma(m\alpha + 1)} = F + \lambda H. \qquad (17)$$

Equations (13) show that both first Caputo derivatives are absolutely continuous. Applying $D_\alpha - \lambda$ twice proves (11) for both functions. Their continuous sequential traces are

$$\begin{pmatrix} F(0) & H(0) \\ D_\alpha F(0) & D_\alpha H(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}. \qquad (18)$$

The determinant is one, proving linear independence. Formula (14) matches both initial data.

For uniqueness, write $U = (y, D_\alpha y)^{\mathsf{T}}$ and $A = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix}$. The identity $I^\alpha D_\alpha u = u - u(0)$ gives $U = U(0) + I^\alpha(AU)$. If two solutions have the same data, their difference Z satisfies, after n iterations,

$$\| Z(x) \| \le \| Z \|_\infty \frac{\| A \|^n T^{n\alpha}}{\Gamma(n\alpha + 1)} \to 0. \qquad (19)$$

Hence $Z = 0$. This proves uniqueness without assuming the existence of a global Laplace transform for every member of the local solution class. □

4.2. Factorization and transform derivation

Set $q = (D_\alpha - \lambda)y$. Equation (11) becomes $D_\alpha q = \lambda q$, with $q(0) = g_0 - \lambda y_0$. Therefore $q = (g_0 - \lambda y_0)F$, and (13) gives (14). Where F is nonzero, this is also the specialization of Proposition 2: since $g = \lambda F$, linearity gives $\mathcal{K}_g = \lambda \mathcal{K}_F$, so (9) reduces to $(D_\alpha - \lambda)\mathcal{K}_F w = 0$. The factorized construction itself does not require division by F.

For the explicitly constructed exponentially bounded functions, sequential Laplace transformation gives

$$\mathcal{L}\{S_\alpha y\} = s^{2\alpha} Y - s^{2\alpha-1} y_0 - s^{\alpha-1} g_0. \qquad (20)$$

Factoring the transformed denominator produces

$$Y(s) = y_0 \frac{s^{\alpha-1}}{s^\alpha - \lambda} + (g_0 - \lambda y_0) \frac{s^{\alpha-1}}{(s^\alpha - \lambda)^2}. \qquad (21)$$

Differentiating the transform of F_λ with respect to λ identifies the second term as the transform of $(g_0 - \lambda y_0)H$. The direct proof above establishes the solution independently of this derivation.

4.3. Classical limit and established matrix structure

For every finite T, as $\alpha \to 1^-$,

$$F(x) \to e^{\lambda x}, \qquad H(x) \to x e^{\lambda x}. \qquad (22)$$

To prove this, restrict α to $[a,1]$ with $a > 0$. For sufficiently large n, monotonicity of the gamma function gives $\Gamma(n\alpha + 1) \ge \Gamma(na + 1)$. On bounded sets in the argument, the defining series and its differentiated series therefore admit summable majorants; the finitely many remaining terms are bounded separately. Termwise convergence yields $E_\alpha(z) \to e^z$ and $E'_\alpha(z) \to e^z$ uniformly on compact sets. Finally, $(\alpha, x) \mapsto x^\alpha$, with $0^\alpha = 0$, is continuous on $[a,1] \times [0,T]$ and hence uniformly continuous. This proves $x^\alpha \to x$ uniformly, completing the argument.

The repeated-root pair is consistent with the usual Jordan-block evaluation of a matrix function. For $J = \lambda I + N$, $N^2 = 0$, the Mittag-Leffler series gives

$$E_\alpha(x^\alpha J) = F(x) I + H(x) N. \qquad (23)$$

Matrix Mittag-Leffler functions already provide fundamental matrices for constant-coefficient Caputo systems (Garrappa and Popolizio, 2018). Thus (12) is an explicit instance of established theory. More generally, the normalized spectral derivatives $\phi_k = (\partial_\lambda^k F_\lambda)/k!$ satisfy $(D_\alpha - \lambda)\phi_k = \phi_{k-1}$ for $k \ge 1$.

Repeated roots of higher multiplicity are therefore not claimed here as an unresolved extension.

5. NUMERICAL EXAMPLE AND REPRODUCIBILITY

Take $\alpha = 2/3$, $\lambda = -1$, $b = 2$, $c = 1$, and initial data $y_0 = 1$, $g_0 = 0$. Formula (14) becomes

$$y(x) = E_{2/3}\left(-x^{2/3}\right) + x^{2/3}E'_{2/3}\left(-x^{2/3}\right). \qquad (24)$$

Table 1 reports the two basis functions and their sum. Its classical comparison column is $e^{-x}(1 + x)$; it is not an exact reference solution at fractional order $2/3$. The values were recomputed from the defining series using 70-decimal arithmetic and a term cutoff of 10^{-55}. The reported digits agree with the supplied 25-decimal computation. This precision comparison is an empirical numerical check, not a certified remainder bound. For demanding arguments, established inversion-based methods are preferable to unguarded direct summation (Garrappa, 2015).

Table 1. Basis functions and solution at α = 2/3; the last column is the classical comparison.

x	F(x)	H(x)	y(x)	Classical
0.5	0.541148	0.290392	0.831541	0.909796
1.0	0.404097	0.295026	0.699122	0.735759
2.0	0.275880	0.252770	0.528650	0.406006
4.0	0.174910	0.182338	0.357248	0.091578
8.0	0.106641	0.115898	0.222539	0.003019

At $x = 1$, Table 2 isolates the limit in (22). It also shows why convergence should not be described as monotone in α over the whole range: the absolute deviation at 0.7 is slightly greater than at 0.5. Figure 1 displays the solution family and the classical curve.

Table 2. Approach to 2/e at x = 1.

α	y(1)	Absolute deviation
0.5	0.700795591	3.4963e-02
0.7	0.700173902	3.5585e-02
0.9	0.718453575	1.7305e-02
0.99	0.733792071	1.9668e-03
0.999	0.735559945	1.9894e-04

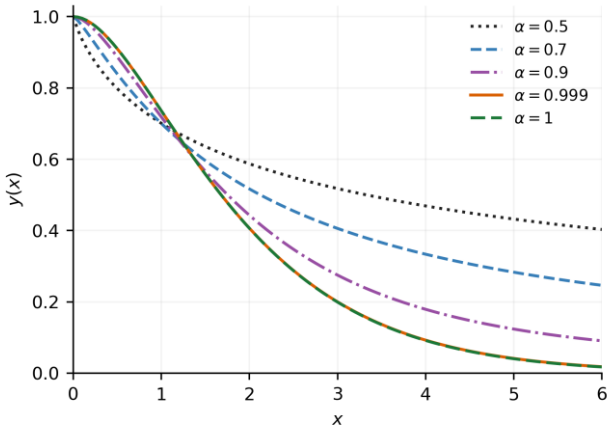


Figure 1. Repeated-root solutions y = F + H for λ = −1 and several α, compared with $e^{-x}(1 + x)$.

5.1. L1 consistency experiments

For $x_n = nh$, define $a_j = (j + 1)^{1-\alpha} - j^{1-\alpha}$. The L1 approximation used here is

$$\delta_h^\alpha u_n = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{n-1} a_j \left(u_{n-j} - u_{n-j-1}\right). \qquad (25)$$

The classical smooth-data error analysis for this scheme assumes regularity that the fractional power profile need not possess (Lin and Xu, 2007). We therefore report residuals and empirical refinement behavior without asserting a general convergence rate.

Two experiments are distinguished. In the identity-assisted experiment, the outer derivative is approximated from samples of the exact function $g = D_\alpha H = F - H$. Its residual at $x = 1$ is $\delta_h^\alpha g_N + 2\delta_h^\alpha H_N + H(1)$. This uses (13) as an input and is a consistency check, not an independent validation of that identity.

In the nested experiment, set $G_0 = 1$, the right trace $D_\alpha H(0)$ from (18), and $G_n = \delta_h^\alpha H_n$ for $n \geq 1$. Approximate the outer derivative by $\delta_h^\alpha G_N$. The residual is then $\delta_h^\alpha G_N + 2G_N + H(1)$. This calculation does not substitute the closed-form inner derivative on the positive grid points, but still requires the correct sequential initial trace. Setting $G_0 = 0$ would represent a different initialization.

Table 3. L1 residuals at x = 1 for the second basis function H.

Experiment	N	Residual
Identity-assisted	100	-9.32651037e-05
Identity-assisted	400	-1.47134824e-05
Identity-assisted	1600	-2.31959926e-06
Nested	100	-1.52409589e-03
Nested	200	-7.33422268e-04
Nested	400	-3.55389315e-04

The first experiment gives residual ratios 6.3388 and 6.3431 under fourfold refinement, corresponding to empirical orders 1.3321 and 1.3326. The nested residual magnitudes decrease by factors about 2.08 and 2.06 under doubling. These observations are restricted to the stated example and evaluation point. Figure 2 summarizes both residual sequences. The analytical proof in Section 4, rather than the numerical experiment, establishes the exact solution.

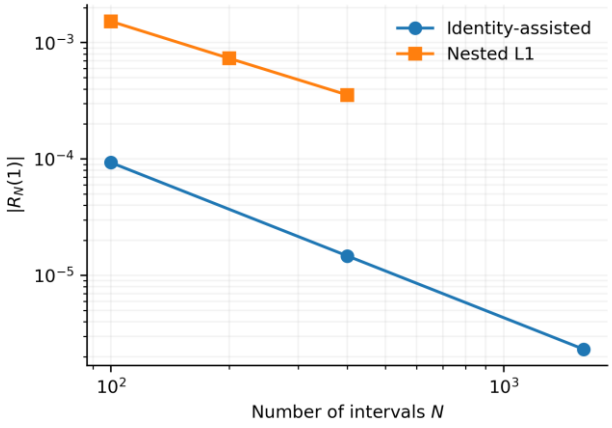


Figure 2. Absolute residuals in the two L1 experiments. The connecting lines do not assert a general error law.

6. DISCUSSION

The distinction between local and nonlocal reduction is central. A fixed finite collection of endpoint derivatives cannot reconstruct a Caputo derivative on the analytic class. Nevertheless, the multiplicative substitution does remove the additive constant in the multiplier: equations (7)–(9) express its remaining influence through v' and memory integrals. These statements are compatible, and neither an infinite formal expansion nor a singular boundary term proves that every order-reduction method fails.

The derivation of (9) is conditional on the stated absolute-continuity class. It does not establish invertibility of $\mathcal{K}_f$, a complete solvability theory for the transformed equation, or a numerical advantage over the original system. For a zero-free known solution it provides an exact representation of the whole solution class on a fixed interval. The repeated-root constant-coefficient problem is simpler because its factorization directly produces a first-order Caputo equation for q.

Jornet (2024) treats sequential equations for the L-fractional derivative and obtains a related repeated-root formula using a different Mittag-Leffler-type function. The normalization is $D_L^{\alpha}u(x) = \Gamma(2-\alpha)x^{\alpha-1}D_{\alpha}u(x)$ for $x > 0$. This factor depends on x and remains inside the next derivative when the operator is composed. Constant-coefficient sequential equations for the two operators therefore cannot be identified merely by replacing their special functions. The parallel formulas illustrate a shared spectral-derivative structure, but do not establish operator equivalence or novelty of the Caputo pair.

Further work may investigate solvability and discretization of (9) for specified variable coefficients, treatment of intervals containing zeros of the known solution, and error estimates for the nested L1 experiment with nonzero sequential traces. Those tasks require additional analysis and are not resolved by the finite-grid residuals reported here.

7. CONCLUSION

For sequential Caputo equations, the classical finite local product-rule mechanism fails, but elimination of the multiplier's additive constant remains possible through an exact nonlocal identity. We established these two facts separately, derived the transformed equation in v', and connected its constant-coefficient repeated-root specialization to operator factorization. The fundamental pair $E_{\alpha}(\lambda x^{\alpha})$ and $x^{\alpha}E'_{\alpha}(\lambda x^{\alpha})$ was verified with explicit regularity, sequential traces, uniqueness, and a compact-uniform classical limit. Its place within existing matrix Mittag-Leffler theory was stated explicitly.

Reproducible numerical examples illustrate the formulas and distinguish identity-assisted from genuinely nested discrete differentiation.

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Data and code availability: The supplementary reproducibility package contains the Python script, numerical tables in CSV format, plotting script, and instructions. No experimental dataset was used. The calculations require Python and mpmath; the figures additionally require NumPy and Matplotlib.

Use of AI tools: An AI assistant supported mathematical checking, literature searches, revision of proofs and presentation, numerical verification, and formatting. The author retains responsibility for the submitted mathematics, references, and conclusions.

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