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Representation of Lie Groups and Algebras with Applications in Differential Equations and Harmonic Oscillators.

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ABSTRACT

This paper presents the representation of Lie groups and Lie algebras with precisely two areas of its applications. We starting on the level of explaining the basic concept of Lie groups, which we then review some connections between Lie groups and Lie algebras and then to their representation and we finally considered its applications in differential equations and harmonic oscillators and currents.

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I. Background of the study

Lie group theory has become more and more pervasive in its influence on other mathematical disciplines. The original founder of this theory was a Norwegian, Marius Sophus Lie who was born in Non reformed, 1842. On the early history of Lie groups, Sophus Lie himself considered the winter of 1873–1874 [13] as the birth date of his theory of continuous groups. Hawkins, however, suggests that it was Lie's prodigious research activity during the four-year period from the fall of 1869 to the fall of 1873 that led to the theory's creation. Some of Lie's early ideas were developed in close collaboration with Felix Klein. Lie meet with Klein every day from October 1869 through 1872 in Berlin from the end of October 1869 to the end of February 1870, and in Paris, Göttingen and Erlangen in the subsequent two years. Lie stated that all of the principal results were obtained by 1884. But during the 1870s all his papers (except the very first note) were published in Norwegian journals, which impeded recognition of the work throughout the rest of Europe. In 1884 a young German mathematician Friedrich Engel, came to work with Lie on a systematic treatise to expose his theory of continuous groups. From this effort resulted the three-volume Theories *der* Transformations gruppen, published in 1888, 1890, and 1893. The term groups *de* Lie first appeared in French in 1893 in the thesis of Lie's student Arthur Tresses. [8]

Lie groups is an intersection of two fundamental fields of mathematics. i.e. algebra and geometry. Lie groups is a first of all the group and secondly it is a smooth manifold which is a specific kind of geometric objects. The circle and the sphere are example of smooth

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manifolds. A circle has a continuous group of symmetries. You can rotate the circle an arbitrarily small amount and it looks the same. Finally, we can say that a Lie group is a group of symmetries where the symmetries are continuous. [13]

Roughly speaking, a Lie group is an analytic manifold with a group structure such that the group operations are analytic. Lie groups arise in a natural way as transformation groups of geometric objects. Lie groups are not linear they are curved manifolds. Nevertheless, Lie's theorem reduces many questions about Lie groups to questions about Lie algebras. Questions about curved manifolds turn out to be equivalent to questions about linear algebra. This is a profound simplification, and it leads to a very rich theory. The theory of Lie groups answers these questions by replacing the notion of a finitely generated group with that of a Lie group a group which at the same time is a finite-dimensional manifold. It turns out that in many ways such groups can be described and studied as easily as finitely generated groups or even easier. Lie groups and Lie algebras, together called Lie theory, originated in the study of natural symmetries of solutions of differential equations. However, unlike say the finite collection of symmetries of the hexagon, these symmetries occurred in continuous families, just as the rotational symmetries of the plane form a continuous family isomorphic to the unit circle. The theory as we know it today began with the ground-breaking work of the Norwegian mathematician Sophus Lie, who introduced the notion of continuous transformation groups and showed the crucial role that Lie algebras play in their classification and representation theory (which one can check [8]). Lie's ideas played a central role in Felix Klein's grand "Erlangen program" to classify all possible geometries using group theory. Today Lie theory plays an important role in almost every branch of pure and applied mathematics, is used to describe much of modern physics, in particular classical and quantum mechanics, and is an active area of research. [12]

II. PRELIMINARIES

Definition 1.2.1: Group is a set G together with a map
$\mu : G \times G \to G; (x, y) \to xy$
and an element $e \to e_G$; such that the following conditions are
fulfilled

i. An associative algebra is algebra A whose associative rule is associative: $x(yz) = (xy)z$ for all $x, y, z \in G$.
ii. $\exists e \in G$, such that for all $x \in G$ we have $x * e = e * x = x$. Such an element $e \in G$ is called an identity in G.
iii. For every $x \in G$ there exists an element $x^{-1} \in G$ such that $x * x^{-1} = x^{-1}x = x$ is called an inverse in G. **[8]**

Definition 1.2.2: A nonempty subset H of a group G is a subgroup of G if H is a group under the same operation as G. We use the notation $H \subset G$ to mean that H is a subset of G, and $H \leq G$ to mean that H is a subgroup of G. For a group G with identity element $e, \{e\}$ is a subgroup of G called the trivial subgroup. For any group G, G itself is a subgroup of G, called the improper subgroup. Any other subgroup of G beside the two above is called a nontrivial proper subgroup of G. **[12]**

Definition 1.2.3: A real (or complex) vector space Ǥ is a real (or complex) Lie algebra, if it is equipped with an additional mapping $[a, a] = 0\ [\cdot,\cdot]$: Ǥ × Ǥ → Ǥ, which is called the Lie bracket and satisfies the following properties:

i. Bilinearity: $[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c]$ and $[c, \alpha a + \beta b] = \alpha[c, a] + \beta[c, b]$ for all $a, b, c \in$ Ǥ and $\alpha, \beta \in F$,
ii. $[a, a] = 0$ for all $a \in$ Ǥ,

Jacobi identity:
$[a,[b,c]] + [b,[c,a]] + [b,[c,a]] + [c,[a,b]] = 0$ .for all $a, b, c \in$ Ǥ. **[8]**

Definition 1.2.4: Let G be a group,

• If G is also a smooth (real) manifold, and the mappings $(a,b)\mapsto ab$ and $a\mapsto a^{-1}$ are smooth, G is a real Lie group,

• If G is also a complex analytic manifold, and the mappings $(a,b)\mapsto ab$ and $a\mapsto a^{-1}$ are analytic, G is a complex Lie group. **[8]**

Definition 1.2.5: Let Ɡ be a Lie algebra.

• Ɡ is semisimple, if there are no nonzero solvable ideals in Ɡ,

• Ɡ is simple, if it is non-Abelian, and contains 0 and Ɡ as the only ideals. **[8]**

Definition 1.2.6: A subgroup H of a Lie group G is called a Lie subgroup if it is a Lie group (with respect to the induced group operation), and the inclusion map $l_H : H \to G$ is a smooth immersion (and therefore a Lie group homomorphism). **[12]**

Definition 1.2.7: A Lie subgroup H of G is said to be a closed Lie subgroup if H is both a Lie subgroup and also a sub manifold of G. **[12]**

Definition 1.2.8: Let G be a real or complex Lie group, then the exponential map $\exp : T_eG \to G$ is define by $\exp(x) = \alpha_x(t)$ Where $\alpha_x(t)$ is the one-parameter subgroup with tangent vector at 1 to x. **[12]**

Definition 1.2.9: A representation of a group G on a vector space V is define as a homomorphism $\phi : G \to GL(V)$. To each $g \in G$, the representation map assigns a linear map, $\rho_g : V \to V$. Although V is actually the representation space, one may for short refer to V as the representation of G. **[8]**

Definition 1.3.0: Harmonic oscillator is a physical system in which some value oscillates above and below a mean value at one or more characteristic frequencies. **[15]**

Definition (1.3.1): A differential Equation is an equation that in addition to independent and dependent variables also contains one or more derivatives of the dependent variables. Thus if y is the dependent variable and it is a function of the independent variable x. These equations are categorized as follows:

(i) If the unknown function depends on a single independent variable, the differential equation is called an Ordinary Differential Equation,

(ii) If it involves partial derivatives with respect to more than one independent variable, the differential equation is called a partial differential equation. A differential equation can be linear or non-linear, first or higher order and initial or boundary value problem. **[19]**

III. REPRESENTATION THEORY

In this section, we shall briefly introduce representations of Lie algebras. For simplicity, we will assume that ḡ is a complex Lie algebra and V a finite dimensional vector space. A representation of ḡ is a homomorphism of Lie algebras $\rho :$ ḡ$\to$ḡ$\iota(V)$.

If ḡ is the Lie algebra of a Lie group, then consider the lemma below

Lemma: The map ad preserves bracket i.e.

$ad([X,Y]_G) = [ad(X), ad(Y)]_E$ Where

$[A,B]_E = AB - BA$ is the bracket on End (ḡ).

Now, we can that ad is a representation of ḡ.

A. The Lie Algebra of a Lie Group

For $x \in G$ consider the conjugation map $C_x = l_x$ or x^{-1}. Since C_x is a homomorphism. Which if we let $(G_, .)$ and $(H_, *)$ group. Function $f : G \to H$ is a homomorphism if for all the element $x, y \in G$ is valid:

i. $f(e_G) = e_H$
ii. $f(xy) = f(x) * f(y)$ for all $x, y \in H$

Proof: If $x \in G$, $f(x) = f(xe_G) = f(x)f(e_G)$ and H is a group, hitting the left with $f(x)^{-1}$, $f(e_G) = e_H$, $f(x)^{-1} = f(x^{-1})$.

Thus, the conjugal map $C_x = l_x$ or x^{-1}. Since C_x is a homomorphism

$$Ad(x) = T_eC_x$$

Is a Lie Algebra Homomorphism.

The map $Ad : G \to GL(T_eG)$ is called the ***adjoint representation*** of G in T_eG.

For $X \in T_eG$ let $adX : T_eG \to T_eG$ be defined by $ad_X(Y) = [X,Y]$. The Jacobi identity is equivalent to saying that

$ad_{[X,Y]} = ad_X ad_Y - ad_Y ad_X = [ad_X, ad_Y]$ i.e.

$ad : T_eG \to End(T_eG)$

Is defined by

$ad = T_e Ad$

We note that, by chain rule, for all $X \in T_eG$;

$$ad(X) = \frac{d}{dt}\Big|_{t=0} Ad(\exp tX)$$

Proposition. If $g \to T_eG$ is a Lie algebra, then

$$ad_{[X,Y]} = ad_X ad_Y - ad_Y ad_X = [ad_X, ad_Y]$$

That is, $ad : T_eG \to End(T_eG)$ is a Lie Homomorphism.

Proof. Observe that

$$ad_{[X,Y]}(Z) = [[X,Y],Z]$$

Whereas

$$[ad_x, ad_y](Z) = [X,[Y,Z]] - [Y,[X,Z]]$$

Thus, we want to show that

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]]$$

Which is equivalent to the Jacobi identity.

B. Connected Lie groups of a given Lie algebra

Let Ꞡ be a finite dimensional Lie algebra. Then there exists a unique (up to isomorphism) connected and simply connected Lie group G with g as its Lie algebra. If G' is another connected Lie group with this Lie algebra, it is of the form G/Z where Z is some discrete central subgroup of G.

We now finally have the whole picture between the connection between Lie groups and Lie algebras (Discussed in [8]). Basically, we now know that we can work with Lie algebras, but thus losing (only) the topology of the group. By knowing possible Lie algebras, we know the possible Lie groups by the following line of thought each Lie algebra Ꞡ generates a unique connected and simply connected Lie group G. Then we also have connected groups of the form G/Z where Z is a discrete central subgroup (central means that it lies in the center of G which is the set of all elements a for which $ab = ba$ for all $b \in G$. We also have disconnected groups with algebra Ꞡ. But they are just the previous groups G/Z overlayed with another (unrelated) discrete group structure.

Before moving on to matrix groups, it is best to look at an example of what we've said so far. We assume some familiarity with matrix groups for this example to make sense. It is a well-known fact that groups $SU(2)$ and $SO(3)$ have the same Lie algebra $\mathfrak{su}(3) = \mathfrak{so}(3)$ Since $SU(2)$ is connected and simply connected, it is the unique group constructed from the Lie algebra $\mathfrak{su}(2)$ And since $SO(3)$ is connected, it means it is isomorphic to $SU(2)/Z$ for some central subgroup Z. It turns out that $SO(3) \cong SU(2)/Z_2$ since $SU(2)$ has the topology of a three dimensional sphere S^3, the quotient group has the topology of the sphere with opposite points identified, which is the real projective space RP^3.

We now come to somewhat more familiar territory. We will consider Lie groups and Lie algebras of matrices.

We define the $GL(n,F)$ as the group of all invertible $n \times n$ matrices, which have either real $(F = R)$ of complex $(F = C)$ entries. Multiplication in this group is defined by the usual multiplication of matrices. The manifold structure is automatic since it is an open set of all $n \times n$ matrices (which form a n^2 dimensional vector space which is isomorphic to F^{n^2}).

We also define Ꞡ$l(n,F)$ as the set of all matrices of dimension $n \times n$ with entries in F. This set is of course a vector space under the usual addition of matrices and scalar multiplication. One can also define the commutator of two such matrices as $[A,B] = AB - BA$ and this operation satisfies the requirements for the Lie bracket. The set Ꞡ$l(n,F)$ therefore has the structure of a Lie algebra.

The notation for Ꞡ$l(nF)$ was suggestive. The matrices Ꞡ $l(n,F)$ are the Lie algebra of the Lie group $GL(n,F)$ including the Lie bracket being the common commutator. The exponential map, which we were unwilling to define in all generality is in this case given as the usual exponential of matrices

$$\exp(A) = e^{A} = I + \sum_{n=1}^{\infty} \frac{A^{n}}{n!}$$

This map can be inverted near the identity matrix I

$$\exp^{-1}(I + A) = \log(I + A) = \sum_{n=1}^{\infty} \frac{(-1)^{k+1} A^{n}}{n}$$

With this definition of the exponential map, we can easily see that for an arbitrary matrix $A, e^{A} \in GL(n, F)$ really is invertible and its inverse is given by e^{-A} Indeed, because $[A - A] = 0$ we have $e^{A}e^{-A} = e^{A-A} = e^{0} = 1$.

Now by virtue of the first fundamental theorem, we can construct various matrix subgroups of $GL(n, F)$ by taking Lie subalgebras of $Gl(n, F)$ namely subalgebras of $n \times n$ matrices. There are a number of important groups and algebras of this type, and they are called the classical groups. We will for the purposes of future convenience and reference list them in a large table together with the restrictions by which they were obtained, as well as some other properties. These properties will be their null and first homotopic group, π^{0} and π^{1}. We will not go further into these concepts here, let us just mention that a trivial π^{0} means G is connected and a trivial while π^{1} means G is simply connected. Furthermore, for G which are not connected π^{1} is specified for the connected component of the identity. Another property which we will also list is whether a group is compact (as a topological space) and denote this by a C. Finally, dim will be the dimension of the group as a manifold which is equal to the dimension of the Lie algebra as a vector space. It is easy to check the dimensionality in each case by noting that $n \times n$ matrices form a n^{2} dimensional space by themselves, but then the dimensionality is gradually reduced by the constraints on its Lie algebra. However, the constraints on the Lie algebra are derived from the constraints on the group. In the orthogonal case for example we have $e^{A}(e^{A})^{T} = (e^{A})^{T} e^{A}$ which implies $e^{A}e^{(AT)} = e^{(A+A^{T})} = I$ and consequently $A + A^{T}$. The constraint $\det e^{A} = 1$ can be reduced to the constraint $TrA = 0$.

IV. APPLICATIONS OF LIE GROUPS AND LIE ALGEBRAS

A. Differential Equations

It may seem odd to speak of the application of Lie algebras and Lie groups to differential equations as a contemporary field of endeavor, but resent work has dealt with quite different problems and techniques from those studied by Lie. Since Lie's work is not widely known, we present a brief indication of his methods before turning to the contemporary applications. For more exposition of this very area of study, one can check Lie's works in *"An introduction of the Lie Theory of parameter Groups, Stechert, New York, 1931."* [2]

Now, suppose we are presented with a first order differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

(i)

And suppose further that we know a one-dimensional Lie group of transformation of the $(x, y)-$ plane which acts analytically and leaves the solutions of the differential equation invariant. With the idea and knowledge of the action of group on the plane, aid us in finding this very solution of the differential equation.

Consider the path or orbit of a given point as the group of transformation acts, i.e. the group acts analytically which means that the x and y coordinates along any orbit are analytic function of the local coordinates in the group. In saying that, the solution is invariant which also means that under the action of a member of the group each point of a given solution curve is carried into a point of another fixed solution curve. Lie was able to show that knowing the orbits one can set up canonical coordinate (u, v) in the plane such that, the action of the group leaves u invariant and such that v is additive in the sense that if the point (u, v) is carried by the group element g_{1} to $(u, v + \delta_{1})$ and by g_{2} to $(u, v + \delta_{2})$ then $g_{2} \circ g_{1}$ carries it to $(u, v + \delta_{1} + \delta_{2})$. In these coordinates, Lie was able showed that the variables separate, so the differential equation can be solved a quadrature. Later, he was able to give an integrating factor (IF) in terms of the tangent vectors to the orbits. If $P(x, y)$ and $Q(x, y)$ are component of the tangent vector to an orbit through (x, y) then $1/(PM + QN)$ is an integrating factor (IF) for the differential equation, and the solution is again

obtained by quadrature. If one knows two distinct groups of transformations (or a two-dimensional group) leaving the differential equation invariant, the quadrature can be eliminated for the solutions are obtained immediately by setting the ratio of the two integrating factors equal to an arbitrary constant.
If we associate with the group, the operator

$$U = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} \quad \text{(ii)}$$

And define an operator associated with the differential equation by

$$A + N\frac{\partial}{\partial x} - M\frac{\partial}{\partial y} \quad \text{(iii)}$$

Then a necessary and sufficient condition for the differential equation to be in-variant under the group is

$$[U, A] = UA - AU = \lambda(x, y)A$$

(iv)

While every first order differential equation can be invariant under the same one-dimentional Lie group, this is not the case for higher order differential equations. When a higher order differential equation is invariant under a one-dimensional group of transformation, however, Lie was able to give a process of introducing new coordinate which reduces the order of differential equation by one. Essentially, one needs to find nontrivial functions $u(x, y), v(x, y, dy/dx)$ which are invariant under the group. When the differential equation is reformulated in terms of u and v it is of order one lower than that of the original equation (Repetition of this procedure reduces the problem to that of solving a first order differential equation). If the dimension of the Lie group under which the differential equation remains invariant is greater than one, the integration procedure can be simplified by introducing canonical coordinates, these canonical coordinates and the integration procedure depend in general on the structure of the Lie algebra of the group of transformations [3] and on the relative dimensions of the group and the orbits. (For instance the group $O(3)$ is a three- dimensional group of transformation on three-dimensional Euclidean space while its nontrivial orbits are the spheres and are two-dimensional).

In addition to giving methods of reducing differential equations invariant under a Lie group to quadrature, Lie was able to determine the differential equation which would be invariant under a giving group. This procedure has been used to advantage in the current work of pattern recognition and most recently, the theory of Lie algebras has been applied to the study of equation of the type

$$\frac{dx}{dt} = \mathbf{A}(t)x \quad \text{(v)}$$

Where **x** and **A** operators and **x** (0) = **I,** the identity which we believe that this problem has a formal solution.

$$\mathbf{x} = \exp\,\Omega(t), \quad \text{(vi)}$$

which actually converges for some interval about $t = 0$ if the operators are matrices and **A** is continuous function of t
let's consider another problem

$$\frac{dx}{dt} = U(t, x) \quad \text{(vii)}$$

This can be regarded as the problem of studying a flow with variable velocity field. An infinitely vector field can be regarded as an operator on the infinitely differentiable scalar functions by identifying its direction with the directional derivative. Thus the vector field $U(t, x)$ would be identified with its dot product $U \bullet \nabla$ with the gradient operator. The set of such operators forms a Lie algebra under commutation, and therefore by this identification we can also regard the victor field themselves as forming a Lie algebra. Now a Lie algebra generated in this manner by the velocity field is finite dimensional, then the Levi structure theorem can be used to obtain the type of decomposition which Wichmann exhibited for some special problems.

B. Harmonic Oscillators and Current

Harmonic oscillators play a fundamental role in quantum mechanics mainly because of the harmonic oscillators underlies the formal apparatus of second quantization. Lie algebras arise naturally as current in quantum field theory.
Let's briefly discuss harmonic oscillators first, the unitary groups arise naturally in the discussion of harmonic oscillators. We adopt the quantum mechanical point of view because it is simpler, but the same type of discussion can be given in classical mechanics. The Hamiltonian for

a system of n in identical non-inter-acting harmonic oscillators is

$$H = \sum_{\alpha=1}^{n} \left\{ \frac{p\alpha^2}{2m} + \frac{k}{2} q\alpha^2 \right\} \qquad \text{(viii)}$$

For convenience, we choose units in which $m = k = 1$ as well as $h = 2\pi$.

We define creation and annihilation operators by

$$a_\alpha = \frac{q_\alpha + ip_\alpha}{\sqrt{2}}, a_\alpha{}^* = \frac{q_\alpha - ip_\alpha}{\sqrt{2}} \qquad \text{(ix)}$$

In terms of which the Hamiltonian may be written as

$$H = (const.) + \sum_{\alpha=1}^{n} a_\alpha{}^* a_\alpha \qquad \text{(x)}$$

The creation and annihilation operators satisfy the commutation relations

$$[a_\alpha, a_\beta] = 0, \; [a_\alpha, a_\beta{}^*] = \delta_{\alpha\beta}, \; [a_\alpha{}^*, a_\beta{}^*] = 0,$$

With each other, and

$$[H, a_\alpha{}^*] = a_\alpha{}^*, \; [H, a_\alpha] = -a_\alpha$$

With the Hamiltonian. These latter commutation relations imply that if ψ is an eigenvector on H with eigenvalue E, then $a_\alpha \psi$ is an eigenvector with eigenvalue $E-1$, and $a_\alpha{}^* \psi$ is an eigenvalue $E+1$.

Both the Hamiltonian and the commutation relations are invariant under a unitary transformation,

$$a_\alpha{}' = \sum U_{\alpha\beta} a_\beta$$

(xi)

So $U(n)$ is a symmetry group of the $n-$dimensional harmonic oscillator. The corresponding Lie algebra $u(n)$ is generated by the element $a_\alpha{}^* a_\beta$, which commute with Hamiltonian H.

We now discuss the conservation of currents. In quantum field theory one introduces for each type of particle a set of creation and annihilation operators on a Hilbert space (the Fok space). If ψ is a certain state (vector in Hilbert space), then the state $a_j{}^* \psi$ is another state which differs from the original state by addition of a single particle of type j. Similarly, $a_j \psi$ is the state obtained by removing a single particle of type j.

Let $a_1, \ldots, a_n$ denote a set of annihilation operators; their Hermitian conjugates $a_1{}^*, \ldots, a_n{}^*$ are creation operators. If we limit discussion to half integer spin particle (fermions), then these operators satisfy the anti-commutation relations,

$$[a_i, a_j] = 0, \; [a_i, a_j{}^*] = \delta_{ij}, \; [a_i{}^*, a_j{}^*] = 0,$$

Where $\{x, y\} = xy + yx$. The physical signature of these relations is the Pauli exclusion principal: no two fermions can occupy the same quantum state. Similarly, if the creation and annihilation operators refer to integer spin particles (bosons) then these operators satisfy commutation relations which are identical to the commutation relations written down in the case of a harmonic oscillator.

One can treat the case of fermions and the case of bosons simultaneously. In either case one can verify the commutation relations,

$$[a_i{}^* a_j, a_k{}^* a_l] = \delta_{jk} a_i{}^* a_l - \delta_{il} a_k{}^* a_j$$

These relations show that the elements $a_i{}^* a_j$ always span a Lie algebra, called the Lie algebra of currents. Any operator of the form

$$J = \sum_{i,j} a_i{}^* c_{ij} a_j \qquad \text{(xii)}$$

Where the c_{ij} are complex numbers, will be called a (generalized) current. One reason for this name is that ordinary electric current corresponds to such an operator in quantum field theory. There are however many other currents, for example, the total spine of a many electron system and the total kinetic energy of a system of particles. A current J which does not depend explicitly on the time is said to be a conserved current if it commutes with the Hamiltonian, $[H, J] = 0$. The set of conserved current forms a Lie algebra.

The formula for the current defines a homomorphism from any Lie algebra of $n \times n$ matrices to the Lie algebra of currents. Let us write the definition in matrix notation as $J = a^* C a$. If K is another current, say $K = a^* D a$, then $[J, K] = a^*[C, D]a$ follows directly from the commutation relations written down for $a_i{}^* a_j$ with $a_k{}^* a_l$, so the mapping $h = C \rightarrow a^* C a$ is a homomorphism.

In conclusion, Lie theory have rich structures and that the mathematical theories of Lie groups and Lie algebras are very beautiful. Lie groups also have applications to many different areas such as: in studying differential equations which we discuss earlier, special functions such as Bessel functions, spherical harmonics etc. according to Klein's Erlanger program, the essence of geometry is to study the invariance of groups acting on homogeneous spaces, Lie groups appears naturally in modern geometric theories, e.g. as the isometry group of a compact Riemannian manifold. Etc. Roughly speaking, a Lie group is a group of symmetries where the symmetries varies smoothly. More precisely, a Lie group admits three structures, an algebraic structure as a group, a geometric structure as a topological manifold and a smooth structure so that one can do analysis. and moreover, these three structures are compatible, i.e. they are naturally related to each other. This make Lie groups into important objects as well as tools in mathematics and other related science.

REFRERENCES

[1] A. C. de Silva. Lecture on symplectic geometry. http://www.math.ist.utl.pt/~acannas/Books/lsg.pdf, 2006.

[2] A. Cohen, An introduction to Lie theory of one parameter groups, strechert, New York, 1931.

[3] Amor Hasic'. An introduction to Lie Groups. Article in advances in Linear Algebra & Matrix Theory. 10,35-51. ISSN: 2165-3348. September 2020.

[4] B. C. Hall. Lie groups, Lie algebras, and representations, volume 222 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2003.

[5] Borel, Armand ., Essays in the history of Lie groups and algebraic groups, History of Mathematics, 21, Providence, R.I.: American Mathematical Society, ISBN 978-0-8218-0288-5, MR 1847105, (2001).

[6] Bourbaki, Nicolas., Elements of mathematics: Lie groups and Lie algebras. Chapters 1–3 ISBN 3-540-64242-0, Chapters 4–6 ISBN 3-540-42650-7, Chapters 7–9 ISBN 3-540-43405-4

[7] Fulton, William; Harris, Joe., Representation theory. A first course. Graduate Texts in Mathematics, Readings in Mathematics. 129. New York: Springer-Verlag. doi:10.1007/978-1-4612-0979-9. ISBN 978-0-387-97495-8. MR 1153249. OCLC 246650103., (1991).

[8] G. Dzarma, D. Samaila. (2019).The classification of semi simple Lie algebra with some application, *International Journal of Mathematics Trends and Technology*- volume 65,2231-5373. 12-December 2019.

[9] Gwyn Bellamy. Lie group, Lie algebras and their Representations. Lecture note for the 5M reading course at the university of Glasgow, autumn 2015.

[10] Hall, Brian C., Lie Groups, Lie Algebras, and Representations: An Elementary Introduction, Graduate Texts in Mathematics, 222 (2nd ed.),Springer, doi:10.1007/978-3-319-13467-3, ISBN 978-3319134666, (2015).

[11] Helgason, Sigurdur., Differential geometry, Lie groups, and symmetric spaces, Graduate Studies in Mathematics, 34, Providence, R.I.: American Mathematical Society, doi:10.1090/gsm/034, ISBN 978-0-8218-2848-9, MR 1834454, (2001).

[12] Hasić, A. (2020) An Introduction to Lie Groups. Advances in Linear Algebra & Matrix Theory , 10, 35-51. https://doi.org/10.4236/alamt.2020.103004

[13] J.G. Belinfanti, B. Kolman, H. A. Smith. An introduction to Lie groups and Lie algebras with applications. SIAM Review, Vol. 8, No. 1, 11-46. Jan, 1966.

[14] Lawson L. M., Gabriel Y. H. A., Laure G. (2017). Lewis-Riesenfeld quantization and SU (1,1) coherent states for 2D damped harmonic oscillator. arXiv.1805.02484v1 [math-ph].

[15] Mukesh Kumar Chundary, Dr. S. Biswas. Basic concept of Lie Groups Internal Journal of Mathematics Trend and Technology volume 67, issue 4, 62-66. ISSN: 2231-5373. April 2021.

[16] Rossmann, Wulf., Lie Groups: An Introduction Through Linear Groups, Oxford Graduate Texts in Mathematics, Oxford University Press, ISBN 978-0-19-859683-7, The 2003 reprint corrects several typographical mistakes, (2001).

[17] Steeb, Willi-Hans., Continuous Symmetries, Lie algebras, Differential Equations and Computer Algebra: second edition, World Scientific Publishing, doi:10.1142/6515, ISBN 978-981-270-809-0, MR 2382250, (2007).

[18] Stillwell, John., Naive Lie Theory. Undergraduate Texts in Mathematics. Springer. doi:10.1007/978-0-387-78214-0. ISBN 978-0387782140, (2008).

[19] Skwame, Y., Sunday, J. & Sabo, J. (2018). On the Development of Two-step Implicit Second Derivative Block Methods for the Solution of Initial Value Problems of General Second Order Ordinary Differential Equations. *Journal of scientific and Engineering Research.* 5, 283-290.